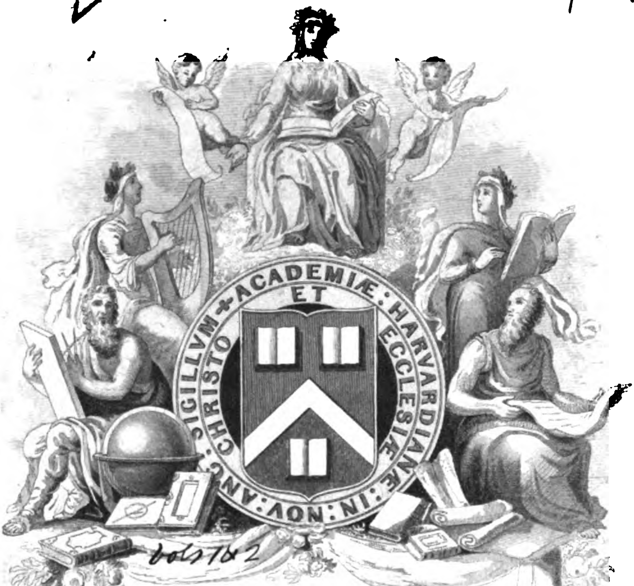

SIGILLVM ACADEMIÆ HARVARDIANÆ IN NOV: ANG:
CHRISTO ET ECCLESIÆ
vols 1&2

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

VOL. II.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστήμη πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

JOHN W. PARKER AND SON, WEST STRAND.

1858.

CONTENTS OF VOLUME II.

No. 5. January, 1857.

THE QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

JOHN W. PARKER AND SON, WEST STRAND.

1857.

W. METCALFE, PRINTER, GREEN STREET, CAMBRIDGE.

PRICE FIVE SHILLINGS.

CONTENTS.

NOTICE TO CORRESPONDENTS.

Papers for the Journal and other communications may be addressed to the Editors under cover to Messrs. J. W. PARKER & SON, 445, *West Strand, London, or to* N. M. FERRERS, Esq., *Gonville and Caius College, Cambridge.*

The next Number will appear in April.

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED MATHEMATICS.

ON THE WAVE SURFACE.

By J. E. Prescott.

Some of the following propositions, although not new, may appear in a form interesting to a portion of your readers. Others are, I think, not known.

1. If $l, m, n, \alpha, \beta, \gamma$, be a direction cosines of the normal to a wave front and of the direction of vibration respectively, we have Fresnel's well-known condition

$$\frac{l}{\alpha}(b^2-c^2)+\frac{m}{\beta}(c^2-a^2)+\frac{n}{\gamma}(a^2-b^2)=0,$$

and since the vibration is in the plane front

$$\frac{l}{\alpha}.\alpha^2+\frac{m}{\beta}.\beta^2+\frac{n}{\gamma}.\gamma^2=0.$$

Eliminating,

$$\frac{l}{\alpha\{\gamma^2(c^2-a^2)+\beta^2(b^2-a^2)\}}=\frac{m}{\beta\{\alpha^2(a^2-b^2)+\gamma^2(c^2-b^2)\}}$$
$$=\frac{n}{\gamma\{\alpha^2(a^2-c^2)+\beta^2(b^2-c^2)\}}.$$

If v be the velocity of propagation, $v^2=a^2\alpha^2+b^2\beta^2+c^2\gamma^2$; and these equations assume the form

$$\frac{l}{\alpha(v^2-a^2)}=\frac{m}{\beta(v^2-b^2)}=\frac{n}{\gamma(v^2-c^2)}=P$$
$$=\frac{\dfrac{l^2}{v^2-a^2}+\dfrac{m^2}{v^2-b^2}+\dfrac{n^2}{v^2-c^2}}{l\alpha+m\beta+n\gamma},$$

and, the denominator being zero, we have also

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0 \;\ldots\ldots\ldots\ldots\; \text{(I)}.$$

2. Now let θ, θ' be the angles between the wave axes and the normal to the front; the direction cosines of the wave cones are $\pm\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)}$, 0, $\pm\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}$. Then

$$\pm\sqrt{(a^2-c^2)}\cos\theta = l\sqrt{(a^2-b^2)}\pm n\sqrt{(b^2-c^2)};$$

$$\therefore\; (a^2-c^2)\sin^2\theta = l^2(b^2-c^2)+m^2(a^2-c^2)+n^2(a^2-b^2)$$
$$\pm 2ln\sqrt{\{(a^2-b^2)(b^2-c^2)\}},$$

and similarly for θ';

$$\begin{aligned}\therefore\; (a^2-c^2)^2\sin^2\theta\sin^2\theta' &= \{l^2(b^2-c^2)+m^2(a^2-c^2)+n^2(a^2-b^2)\}^2\\ &\quad -4l^2n^2(a^2-b^2)(b^2-c^2)\\ &= \{l^2(b^2+c^2)+m^2(a^2+c^2)+n^2(a^2+b^2)\}^2\\ &\quad -4(l^2b^2c^2+m^2a^2c^2+n^2a^2b^2)\\ &= (v_1^2+v_2^2)^2-4v_1^2v_2^2,\end{aligned}$$

where v_1, v_2 are the roots of (I);

$$\therefore\; \pm(a^2-c^2)\sin\theta\sin\theta' = v_1^2-v_2^2.$$

Also

$$\begin{aligned}\pm(a^2-c^2)\cos\theta\cos\theta' &= l^2(a^2-b^2)-n^2(b^2-c^2)\\ &= a^2+c^2-\{l^2(b^2+c^2)+m^2(a^2+c^2)+n^2(a^2+b^2)\}\\ &= a^2+c^2-(v_1^2+v_2^2).\end{aligned}$$

3. If λ, μ, ν be the direction cosines of any radius (r) of the wave surface, we have, from Mr. A. Smith's method of determining its equation, the relations

$$\frac{\lambda(v^2-a^2)}{l(r^2-a^2)}=\frac{\mu(v^2-b^2)}{m(r^2-b^2)}=\frac{\nu(v^2-c^2)}{n(r^2-b^2)}=\frac{v}{r}=\cos\delta=Q,$$

δ evidently being the angle between the ray and the normal to the corresponding front; and

$$\frac{l^2}{(v^2-a^2)^2}+\frac{m^2}{(v^2-b^2)^2}+\frac{n^2}{(v^2-c^2)^2}=\frac{1}{v^2(r^2-v^2)}.$$

Now by (1) $P=\left\{\frac{l^2}{(v^2-a^2)^2}+\frac{m^2}{(v^2-b^2)^2}+\frac{n^2}{(v^2-c^2)^2}\right\}^{\frac{1}{2}}$;

$$\therefore\; P=\frac{1}{v\sqrt{(r^2-v^2)}}.$$

And combining equations (P) and (Q)

$$\frac{\lambda}{\alpha(r^2-a^2)} = \frac{\mu}{\beta(r^2-b^2)} = \frac{\nu}{\gamma(r^2-c^2)} = \frac{1}{r\sqrt{(r^2-v^2)}} = R.$$

The equations (P), (Q), (R) connect respectively the normal and vibration, the normal and ray, the ray and vibration.

4. The equation to the wave surface may be put in the form

$$\frac{\lambda^2}{\frac{1}{r^2}-\frac{1}{a^2}} + \frac{\mu^2}{\frac{1}{r^2}-\frac{1}{b^2}} + \frac{\nu^2}{\frac{1}{r^2}-\frac{1}{c^2}} = 0 \text{ .... (II).}$$

Now if ϕ, ϕ' be the angles between the ray and the ray axes, r_1, r_2 the roots of this equation, by pursuing precisely the same expedient with regard to (II) that we did in (2), with regard to (I) we can prove that

$$\pm\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\sin\phi\,\sin\phi' = \frac{1}{r_1^{\,2}} - \frac{1}{r_2^{\,2}}.$$

This law was discovered experimentally by Biot and Brewster, stated by the former in the language of the theory of emission. It was afterwards proved analytically by Fresnel. r_1, r_2 measuring the velocities of the two waves in the directions of the luminous rays themselves.

5. From (R) we have

cos(angle between ray and direction of vibration)

$$= \lambda\alpha + \mu\beta + \nu\gamma = r\sqrt{(r^2-v^2)}\left(\frac{\lambda^2}{r^2-a^2} + \frac{\mu^2}{r^2-b^2} + \frac{\nu^2}{r^2-c^2}\right)$$
$$= \frac{1}{r}\sqrt{(r^2-v^2)} = \sin S;$$

hence, the angle between the ray and the direction of vibration is the complement of the angle between the ray and the normal to the front. Now the directions of vibration are perpendicular to the normal and to one another; therefore, the plane, containing the ray and its corresponding direction of vibration, passes through the normal to the front. Through the two directions of vibration there will be two planes, at right angles, passing one through each of the two rays determined by (II) and intersecting in the normal to the front. Hence, to find the direction of vibration corresponding to any ray, we have only to project the ray on the tangent plane, at the point where the ray meets the surface. This projection will be the direction of vibration. Also, the planes containing either ray and the directions of vibration will evidently be at right angles.

6. It is easily shewn (Griffin's *Tract*, 31) that the equation to the tangent cone at a singular point of the wave surface is

$$\left\{x\sqrt{(a^2-b^2)}+z\frac{c}{a}\sqrt{(b^2-c^2)}\right\}\left\{x\sqrt{(a^2-b^2)}+z\frac{a}{c}\sqrt{(b^2-c^2)}\right\}$$
$$-\frac{a^2-c^2}{4a^2c^2}(a^2-b^2)(b^2-c^2)y^2=0.$$

Now the luminous cone, along the generating lines of which the rays, producing external conical refraction, pass, will be formed by drawing normals to all the tangent planes at the singular point. To find its equation put, for brevity, the above in the form

$$(Aax+Ccz)(Acx+Caz)=B^2y^2,$$

$$Aax+Ccz=\lambda By,\quad Acx+Caz=\frac{1}{\lambda}By,$$

λ being an arbitrary constant.

Eliminating between these two equations

$$\frac{Ax}{\lambda a-\dfrac{c}{\lambda}}=\frac{By}{a^2-c^2}=\frac{Cz}{\dfrac{a}{\lambda}-c\lambda}.$$

Hence the equation to a tangent plane of the cone required is

$$x\frac{\lambda^2a-c}{A}+y\frac{\lambda(a^2-c^2)}{B}+z\frac{a-\lambda^2c}{C}=0,$$

differentiating this equation with regard to λ

$$\frac{y}{B}(a^2-c^2)+2\left(\frac{ax}{A}-\frac{cz}{C}\right)\lambda=0,$$

and substituting in it

$$\frac{\lambda y}{B}(a^2-c^2)-2\left(\frac{cx}{A}-\frac{az}{C}\right)=0;$$

therefore, the equation to the cone is

$$\frac{y^2}{B^2}(a^2-c^2)+4\left(\frac{ax}{A}-\frac{cz}{C}\right)\left(\frac{cx}{A}-\frac{az}{C}\right)=0,$$

or giving A, B, C their values

$$\left\{x\sqrt{(b^2-c^2)} - z\frac{c}{a}\sqrt{(a^2-b^2)}\right\}\left\{x\sqrt{(b^2-c^2)} - z\frac{a}{c}\sqrt{(a^2-b^2)}\right\}$$
$$+ (a^2-c^2)y^2 = 0 \ldots\ldots\ldots \text{E.}$$

7. To obtain the circular sections of this cone. The projections on the plane of xy of the sections of the cone and of a sphere

$$x^2 + y^2 + z^2 + \ldots\ldots = 0,$$

by a plane $lx + my + nz = p$, must coincide.

Substituting for z in the two equations, and comparing the coefficients of xy, we see that m must be zero; and the ratio of the coefficients of x^2, y^2 gives

$$n^2(b^2-c^2) + l^2(a^2-b^2) + ln\frac{a^2+c^2}{ac}\sqrt{\{(a^2-b^2)(b^2-c^2)\}} = (l^2+n^2)(a^2-c^2),$$

or $$\frac{l^2}{n^2} - \frac{l}{n}\frac{a^2+c^2}{ac}\sqrt{\left(\frac{a^2-b^2}{b^2-c^2}\right)} + \frac{a^2-b^2}{b^2-c^2} = 0.$$

The roots of this equation determine the planes which give the required circular sections. One root is $\frac{c}{a}\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)}$, the other $\frac{a}{c}\sqrt{\left(\frac{a^2-b^2}{b^2-c^2}\right)}$. Hence the circular sections are parallel to the tangent of the ellipse or circle, in the plane of xz, at the point of their intersection.

8. The directions of vibration, corresponding to the generators of the luminous cone (E), will be determined by projecting the ray axis upon the tangent planes at the singular point (5). We may therefore consider them as all passing through the point where the ray axis meets the wave surface and each point of the curve determined by the intersection of a sphere described on the ray axis (b) as diameter, and the cone (E) moved parallel to itself, so as to have the center of the wave surface for the origin. The equation to the sphere is

$$x^2 + y^2 + z^2 = xc\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)} + za\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)},$$

and that to the cone

$$(b^2-c^2)x^2 + (a^2-c^2)y^2 + (a^2-b^2)z^2 = xz\frac{a^2+c^2}{ac}\sqrt{\{(a^2-b^2)(b^2-c^2)\}},$$

subtracting

$$\left\{xc\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)}+za\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}\right\}\left\{\frac{x}{c}\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)}+\frac{z}{a}\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}-1\right\}=0;$$

$$\therefore\ \frac{cx}{c^2}\sqrt{\left(\frac{a^2-b^2}{a^2-c^2}\right)}+\frac{az}{a^2}\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}=1,$$

which is the equation to a plane, perpendicular to the plane of xz, and passing through the tangent to the ellipse, in the plane of xz, at the angular point. Hence the curve sought is the circle in which the sphere is cut by this plane. The direction will be determined by joining each point of this circle, whose plane is perpendicular to that of xz, to the point where the ray axis meets it. And we may consider the corresponding planes of polarization as perpendicular to the plane of this circle, and joining the corresponding points to that of the circle, which is opposite to the point in which the ray axis meets it, as was originally determined by Sir William Hamilton.

9. Let α, β be the angles which the ray axis and the tangent to the ellipse at the singular point made with the axis of x; x', z' coordinates of the singular point. Then

$$\tan\beta=\frac{a^2}{z'}\cdot\frac{x'}{c^2}=\frac{a}{c}\sqrt{\left(\frac{a^2-b^2}{b^2-c^2}\right)},\quad \tan\alpha=\frac{a}{c}\sqrt{\left(\frac{b^2-c^2}{a^2-b^2}\right)};$$

therefore $\tan(\alpha+\beta)=-\dfrac{ac}{\sqrt{\{(a^2-b^2)(b^2-c)\}}}$.

Hence tangent of angle of tangent cone in the plane of xz

$$=\cos(\alpha+\beta)=-\frac{\sqrt{\{(a^2-b^2)(b^2-c^2)\}}}{ac}.$$

tangent of angle of luminous cone in the plane of xz

$$=-\cot(\alpha+\beta)=\frac{\sqrt{\{(a^2-b^2)(b^2-c^2)}}{ac},$$

diameter of circle of vibrations

$$=b\cos(\alpha+\beta)=\sqrt{\frac{\{(a^2-b^2)(b^2-c^2)\}}{a^2+c^2-b^2}},$$

10. By putting $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ for a, b, c in (E), we get the equation to the luminous cone producing internal conical

refraction,

$$\{x\sqrt{(b^2-c^2)} - z\sqrt{(a^2-b^2)}\}\left\{x\frac{a^2}{b^2}\sqrt{(b^2-c^2)} - z\frac{c^2}{b^2}\sqrt{(a^2-b^2)}\right\}$$
$$+ (a^2-c^2)\,y^2 = 0\ldots\ldots\ldots(F).$$

It is shewn easily (Griffin's *Tract*, 34) that a section of this cone, by a plane perpendicular to the wave axis, is a circle; the section at a distance (b) from the vertex, being the circle of contact; the other circular sections of the cone will evidently be the subcontrary sections, made by a plane perpendicular to the other generator in the plane of xz.

11. The directions of vibration, corresponding to the generators of this luminous cone (F), will be determined by projecting these rays on the corresponding tangent plane to the wave surface, *i.e.* on the plane of the circle of contact, which is perpendicular to the plane of xz. We may therefore consider them as joining each point of this circle to the point where the wave axis meets it, and the planes of polarization, as perpendicular to the plane of this circle and joining the corresponding points to that point of the circle, which is opposite to the point in which the wave axis meets it.

12. Let α', β' be the angles formed with the axis of x by the radii joining the centre of the wave-surface to the points of contact of the tangent, common to the circle and ellipse, in the plane of xz,

$$\tan\beta' = \frac{a^2\sin\alpha'}{b}\cdot\frac{b}{c^2\cos\alpha'} = \frac{a^2}{c^2}\tan\alpha',\quad \tan\alpha' = \sqrt{\left(\frac{b^2-c^2}{a^2-b^2}\right)}.$$

Hence tangent of angle of luminous cone on the plane of xz

$$= \tan(\beta'-\alpha') = \frac{\sqrt{\{(a^2-b^2)(b^2-c^2)\}}}{b^2}.$$

Diameter of circle of vibrations

$$= b\tan(\beta'-\alpha') = \frac{\sqrt{\{(a^2-b^2)(b^2-c^2)\}}}{b}.$$

13. By turning the coordinate axes, so that the axis of x coincides first with the ray and then with the wave axis, the equations to the cones (E) and (F) assume the simple forms,

$$y^2 + z^2 + zx\,\frac{\sqrt{\{(a^2-b^2)(b^2-c^2)\}}}{ac} = 0,$$

and
$$y^2 + z^2 - zx\,\frac{\sqrt{\{(a^2-b^2)(b^2-c^2)\}}}{b^2} = 0.$$

From the second we obtain at once, putting $x=b$, the equation to the circle of contact, radius $\dfrac{\sqrt{\{(a^2-b^2)(b^2-c^2)\}}}{2b}$.

14. Professor Lloyd, in performing the experiment of conical refraction, observed the luminous circle with a tourmaline; he then perceived that only one ray disappeared at a time, and that on turning the tourmaline through any angle, another ray disappeared, at a point such that the intervening arc of the circle subtended, at the centre of the circle, an angle double of the angle through which the tourmaline had been turned. Whence he deduced the law, "The angle between the planes of polarization of any two rays of the cone, is half the angle between the planes through the rays and the axis of the cone." Now from (7) and (11) we see that in both cases this law is true; for in either circle the arc, cut off by two planes of polarization, subtends, at the centre of the circle, an angle which is double that between the planes of polarization.

THE PLANETARY THEORY.

By Rev. Percival Frost.

1. The exact paths described by more than two bodies acting mutually upon one another with forces varying according to the Law of Gravitation, and projected in any manner in space, have never been determined. Mathematical Analysis, in its present state, has been unable to supply a solution of the complications of the problem presented to it.

Hence, having failed in their attempts to solve the general problem of the motion of three bodies, mathematicians have been compelled to examine whether any circumstances, which belong peculiarly to the Solar System, would enable them to take such a view of any portions of it, that, by limitations introduced into the more general problem, they might be able to apply the lever of their analysis with advantage sufficient to obtain the position of the planets and their secondaries, at any given time, if not with rigid exactness, at least with sufficient accuracy to compete with the delicacy which is introduced into astronomical observations in the present advanced state of Practical Astronomy.

This has been done, and the peculiar circumstances of which they have availed themselves may be stated as follows:

(1) The Sun, which is the largest body of the system, has a mass which is a thousand times greater than that of the largest of the accompanying bodies whose motion is required, and hundreds of thousands of times greater than that of the smaller ones.

This preponderance would imply that the attraction of the Sun upon any planet must be greater in those proportions than that of the accompanying bodies on the same planet, unless in any case it should happen that this preponderance of mass were compensated for by a greater degree of proximity; but the favourable conditions, under which the bodies of the Solar System are at present moving, prevent any compensation of the kind; for, even when the Sun is at its greatest distance from the planet acted upon, and any of the disturbing planets at their least distance, it is easily shewn that the action of the disturbing planet can only bear a very small ratio, the greatest being less than 1 : 100, to that of the Sun in its most unfavourable position.

(2) The planets, though not the planetoids and comets, are observed to describe paths which differ by extremely small quantities from the ellipses which they would each describe about the Sun, if at any instant the other planets were to cease to exist. In fact, the deviations from such ellipses are so slight that they are only perceptible under the handling of the most delicate methods of measurement, and in many cases even such treatment fails to detect the deviations, except by allowing them to accumulate by the action of the disturbances during a long period.

(3) The eccentricities of the ellipses, which so nearly represent the paths of the planets, are extremely small; so that, if these paths were represented on paper on a large scale, the fact that they are not circles could only be established by very exact measurements of the breadths in different directions.

(4) The inclinations of the planes of these ellipses to one another are very small.

(5) The planets are very nearly, though not exactly, of a spherical form, so that their attractions may be considered the same as if they were collected at their centers of gravity, as far as their actions upon one another are concerned,

although the deviation from perfect sphericity is sufficient to affect the motion of their satellites and of themselves about their centers.

Definition of the Instantaneous Ellipse.

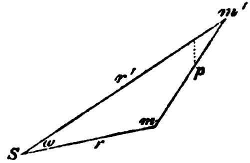
2. The ellipse, which would be described by a planet about the Sun's center, supposed fixed, as one of the foci, if the disturbing forces exerted by the other planets were supposed to cease to act, is called the *Instantaneous Ellipse.*

The elements of the Instantaneous Ellipse are the *mean distance* and *eccentricity*, which determine the magnitude and form of the ellipse; the *longitude of perihelion*, which determines the position of the major axis; the *longitude of the node*, and the *inclination* of the plane of the ellipse to a fixed plane of reference, which determine the position of the ellipse in space; and the mean *longitude of the epoch*, or the mean longitude of the position at which the planet would have been at the fixed epoch from which the time is measured, if it had been moving during that time in the undisturbed orbit; this last element serves according to the ordinary elliptic theory to determine the position of the planet in the instantaneous ellipse at the time under consideration, which position by construction coincides with the actual position of the planet at that time.

3. We shall commence by determining the law of the disturbing forces in the case of one disturbing planet, and afterwards extend the investigation to the case of any number of planets: we shall then investigate the rate of change of the elements of the instantaneous ellipse, and thence obtain formulæ for calculating the elements themselves at any fixed time.

To calculate the disturbing forces exerted by a planet upon another in motion about the Sun.

4. Let S, m, m' be the positions of the Sun, the disturbed and the disturbing planets at any time; r, r' the distances of m and m' from S; ρ that of m from m'. And let S, m, m' be the measures of the accelerations of these bodies at an unit of distance.



The forces on S are $\dfrac{m}{r^2}$ in Sm,

and $\dfrac{m'}{r'^2}$ in Sm'.

The forces on m are $\frac{S}{r^2}$ in mS,

and $\frac{m'}{\rho^2}$ in mm'.

The relative motions will not be disturbed if we apply to every part of the system the forces on the Sun in a contrary direction, in which case the Sun may be considered as at rest, and m as acted on by the forces

$$\frac{S+m}{r^2} = \frac{\mu}{r^2} \text{ in } mS,$$

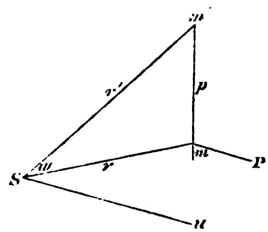
$$\frac{m'}{r'^2} \text{ parallel to } m'S,$$

$$\text{and} \quad \frac{m'}{\rho^2} \text{ in } mm'.$$

Of these forces $\frac{\mu}{r^2}$ is that under the action of which the Instantaneous Ellipse would be described if the action of m' ceased, and the disturbing force on m is the resultant of $\frac{m'}{r'^2}$ and $\frac{m'}{\rho^2}$ acting in the directions determined above.

To investigate an expression for the component of the disturbing forces estimated in any direction.

5. If r, r' be the distances of the planets m, m' from the Sun S, ω the angle between them, ρ the distances of m from m'.



Let Su be any given direction.

The disturbing forces of m' on m are $\frac{m'}{r'^2}$ in the direction parallel to $m'S$, and $\frac{m'}{\rho^2}$ in the direction parallel to mm'.

The resolved parts of these forces in the direction Su are

$$-\frac{m'}{r'^2}\cos m'Su + \frac{m'}{\rho^2}\cos m'mP,$$

where $mP = du$ is a displacement of m in the direction Su.

Then $\qquad du.\cos m'Su = d(r\cos\omega),$

and $\qquad du.\cos m'mP = -d\rho;$

therefore, the component in the direction δu

$$= -\frac{m'}{r'^3}\frac{d(r\cos\omega)}{du} - \frac{m'}{\rho^2}\frac{d\rho}{du}$$

$$= \frac{dR}{du};$$

if $$R = -\frac{m'r\cos\omega}{r'^3} + \frac{m'}{\rho}.$$

This function is called *the disturbing function.*

To obtain measures of the Radial, Transversal, and Orthogonal disturbing forces.

6. The position of a point in space may be determined by the polar coordinates, r, θ of its projection upon the plane of the instantaneous ellipse constructed for any time t and the distance ζ from that plane.

If the planet m be supposed to receive successively the geometrical displacements Δr, $r\Delta\theta$, and $\Delta\zeta$, while m' remains stationary R undergoes a corresponding change, and the limits of corresponding ratios $\frac{\Delta R}{\Delta r}$, $\frac{\Delta R}{r\Delta\theta}$, $\frac{\Delta R}{\Delta\zeta}$, are the components of the disturbing forces in those directions;

therefore, the radial disturbing force $= \frac{dR}{dr}$,

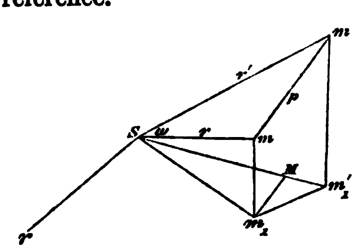
the transversal .... $= \frac{dR}{rd\theta}$,

the orthogonal .... $= \frac{dR}{d\zeta}$,

where each derivative of R is taken with respect to r, θ, ζ as if they were the only variables.

To express R *in a form prepared for development.*

7. Let m_1, m_1' be the projections of m and m' on the fixed plane of reference.



$$Sm_1 = r_1, \qquad Sm_1' = r_1',$$
$$mm_1 = z \qquad m'm_1' = z',$$

θ_1, θ_1' the longitude of m and m';

or
$$\angle\, m_1Sm_1' = \theta_1' - \theta_1,$$

$$\rho^2 = m_1m_1'^2 + \overline{z'-z}|^2$$
$$= r_1^2 - 2r_1r_1' \cos(\theta_1' - \theta) + r_1'^2 + (z'-z)^2,$$

$r\cos\omega =$ projection of Sm on Sm'

$=$ sum of the projections of SM, Mm_1, and m_1m on Sm' (where m_1M is perpendicular to Sm_1')

$$= SM\frac{r_1'}{r'} + z\frac{z'}{r'};$$

therefore
$$R = -\frac{m'\{r_1r_1'\cos(\theta_1'-\theta_1) + zz'\}}{(r_1'^2 + z'^2)^{\frac{3}{2}}} + \frac{m'}{\{r_1^2 - 2r_1r'\cos(\theta_1 - \theta_1) + r_1'^2 + (z'-z)^2\}^{\frac{1}{2}}}.$$

To express R *in terms of the elements of the instantaneous orbits of* m *and* m' *at the time* t.

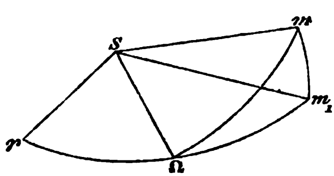
8. Let Ω_1, i_1, ϖ_1, a_1, e_1, n_1, ε_1 be the longitude of the node, the inclination, the longitude of perihelion measured on the plane of reference up to the node and thence on the plane of $m'S$ instantaneous ellipse, the mean distance, the eccentricity of the ellipse, the mean angular velocity, and the mean longitude at the commencement of the epoch, supposing the motion of m undisturbed.

θ the longitude of m on its instantaneous plane. Therefore

$$\theta = n_1t + \varepsilon_1 + 2e_1\sin(n_1t + \varepsilon_1 - \varpi_1) + \frac{5e_1^2}{4}\sin 2(n_1t + \varepsilon_1 - \varpi_1) + \&c.,$$

$$r = a_1\left(1 + \frac{e_1^2}{2} - e_1\cos(n_1t + \varepsilon_1 - \varpi_1) - \frac{e_1^2}{2}\cos 2(n_1t + \varepsilon_1 - \varpi_1) + \&c.\right).$$

Let Ωm be the plane of m's instantaneous orbit, Sm_1 the



projection of Sm, $S\Upsilon$ drawn towards the first point of Aries, $S\Omega$ the line of nodes.

$$\tan(\theta_1 - \Omega_1) = \cos i_1 \tan(\theta - \Omega_1);$$

therefore $$\tan(\theta_1 - \theta) = -\frac{(1-\cos i_1)\tan(\theta-\Omega_1)}{1+\cos i_1 \tan^2(\theta-\Omega_1)}$$

$$= -\frac{(1-\cos i_1)\tan(\theta-\Omega_1)}{\sec^2(\theta-\Omega_1)-(1-\cos i_1)\tan^2(\theta-\Omega_1)}$$

$$= -\sin^2\frac{i_1}{2}\sin 2(\theta-\Omega_1),$$

neglecting squares of $\sin^2\frac{i_1}{2}$;

therefore $$\theta_1 - \theta = -\tan^{-1}\left(\sin^2\frac{i_1}{2}\sin 2(\theta-\Omega_1)\right)$$

$$= -\sin^2\frac{i_1}{2}\sin 2(\theta-\Omega_1) \text{ to the same order;}$$

therefore $$\theta_1 = \theta - \sin^2\frac{i_1}{2}\sin 2(\theta-\Omega_1),$$

and $$r_1 = r\cos mSm_1 = r\left(1-\tfrac{1}{2}\tan^2 mSm_1\right) \text{ nearly}$$

$$= r\left(1-\tfrac{1}{2}\tan^2 i_1 \sin^2(\theta_1-\Omega_1)\right)$$

$$= r - \frac{a_1}{4}\tan^2 i_1 + \frac{a_1}{4}\tan^2 i_1 \cos 2(\theta_1-\Omega_1),$$

$$z = r_1 \tan nSm_1 = r_1 \tan i_1 \sin(\theta_1-\Omega_1),$$

whence, with similar expressions for r', θ', and z', R can be expressed as a function of the elements of the orbits, and the time.

In order to adapt this expression for R to the purposes of numerical calculation, it is necessary to develope it according to ascending powers of the small quantities, the eccentricities of the instantaneous ellipse, and the inclinations of the orbits to the fixed plane of reference.

This is a development accompanied by considerable difficulties; and at present we shall suppose that the development has been effected, and proceed to the explanation of methods of obtaining the values of the elements of the instantaneous ellipse at any time.

To calculate the disturbing forces in the case of more planets than one.

9. If we neglect the squares of the disturbing forces, *i.e.* if we consider the disturbance of a disturbance as too small to be considered, the disturbance of the motion of the disturbed planet from exact elliptical motion in any direction will be the algebraical sum of the disturbances, which would be caused in the direction by the action of each planet, if supposed to be the only disturbing cause.

Hence, if R', R'',...... be the disturbing functions corresponding to the planets m', m'',...... R the disturbing function corresponding to combined action of all the disturbing planets,

$$R = R' + R'' + \ldots\ldots$$

and $\frac{dR}{du}$ is the component, in the direction Su, of the resultant of all the disturbing forces.

General description of the method of determining the rate of change of the elements of the instantaneous ellipse.

10. The elements of the instantaneous ellipse depend upon the velocity and direction of motion of the planet at the observed time: this dependence is expressed by means of equations.

These equations are of the same form in the case of two instantaneous ellipses constructed for any two different instants.

If therefore the equations be prepared, corresponding to the two ellipses constructed for the times t and $t + \delta t$ separated by a very small interval δt, by taking the differences, a new system of equations is found which connect the small changes of the elements with the changes of the velocity and direction of motion of the planet during the interval δt.

The changes of the velocity and direction of motion are due to the action of the principal force and of the disturbing forces, whose accelerating effects are measured by $\frac{\mu}{r^2}$, and the differential coefficients of the disturbing function R; they can therefore be expressed in terms of these measures and the time δt of their action.

Hence, by substitution of these expressions in the new system of equations, the changes of each element can be expressed in the form $U\delta t$, where U is a function of the elements at the time and of t; and U the rate of change is determined.

To calculate the rate of change of the mean distance of the instantaneous ellipse.

11. Let a_1, $a_1 + \delta a_1$ be the mean distances in the instantaneous ellipses constructed for the times t, $t + \delta t$ from a fixed epoch.

r, $r + \delta r$ the radii vectores of the planet,

v^2, $v^2 + \delta v^2$ the squares of the velocities.

$$\text{Then} \quad v^2 = \frac{2\mu}{r} - \frac{\mu}{a_1},$$

$$v^2 + \delta v^2 = \frac{2\mu}{r + \delta r} - \frac{\mu}{a_1 + \delta a_1}.$$

Hence if δt be very small, neglecting the squares of the small changes,

$$\delta v^2 = -\frac{2\mu\delta r}{r^2} + \frac{\mu\delta a_1}{a_1^2} \quad \text{.....................(1).}$$

Now, δv^2 is the change of the square of the velocities due to the action of the forces whose measures are $-\frac{\mu}{r^2} + \frac{dR}{dr}$ in the direction Sm, $\frac{dR}{rd\theta}$ in the direction perpendicular to Sm and tending to increase θ, and $\frac{dR}{d\zeta}$ in the direction perpendicular to the plane of the ellipse for the time t.

Hence if v_1, v_2, v_3 be the velocities in those directions, considering the forces constant during the time δt,

$$\delta v^2 = \delta v_1^2 + \delta v_2^2 + \delta v_3^2$$

$$= 2\left(-\frac{\mu}{r^2} + \frac{dR}{dr}\right)\delta r + 2\frac{dR}{rd\theta} \cdot r\delta\theta + 2\frac{dR}{d\zeta}\,d\zeta;$$

therefore, substituting in equation (1), we obtain

$$\frac{\mu\delta a_1}{a_1^2} = 2\left(\frac{dR}{dr}\delta r + \frac{dR}{d\theta}\delta\theta + \frac{dR}{d\zeta}\delta\zeta\right);$$

and since, at the time t, m has no velocity perpendicular to the plane of the instantaneous ellipse, $\delta\zeta$ is entirely due to the action of the disturbing force and is of the order $\frac{dR}{d\zeta}\delta t^2$, and may therefore be neglected.

Therefore, dividing by δt and proceeding to the limit,

$$\frac{\mu}{a_1^2}\frac{da_1}{dt} = 2\left(\frac{dR}{dr}\frac{dr}{dt} + \frac{dR}{d\theta}\frac{d\theta}{dt}\right)$$

$$= 2\frac{d(R)}{dt} \quad \text{......................... (2),}$$

where $\frac{d(R)}{dt}$ denotes the differential coefficient of R with respect to t only so far as R is a function of r and θ.

If now we refer to the values of r and θ, we shall see that in the differentiation R must be considered as a function of t, only so far as t is involved explicitly in the expression $n_1t + \varepsilon_1$, and implicitly in the elements a_1, e_1, ε_1, n_1, and ϖ_1: and since the differential coefficients of these elements depend upon the disturbing forces, the squares of which are neglected, these elements may be treated as constants; therefore

$$\frac{d(R)}{dt} = \frac{n_1 dR}{d(n_1t + \varepsilon_1)} = n_1\frac{dR}{d\varepsilon_1};$$

therefore, by (2), the rate of change $= \frac{da_1}{dt} = \frac{2n_1a_1^2}{\mu}\frac{dR}{d\varepsilon_1}$.

To calculate the rate of change of the eccentricity of the instantaneous ellipse.

12. Let e_1, $e_1 + \delta e_1$ and h_1, $h_1 + \delta h_1$ be the eccentricities and double sectional areas at the times t, and $t + \delta t$.

Then since $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{dR}{rd\theta}$ by the equations of motion of m,

$$\frac{dh_1}{dt} = \frac{dR}{d\theta},$$

and
$$\theta = n_1t + \varepsilon_1 + f(n_1t + \varepsilon_1 - \varpi_1),$$
$$r = \phi(n_1t + \varepsilon_1 - \varpi_1);$$

therefore
$$\frac{dR}{d\varepsilon_1} = \frac{dR}{dr}\cdot\frac{dr}{d\varepsilon_1} + \frac{dR}{d\theta}\frac{d\theta}{d\varepsilon_1},$$
$$\frac{dR}{d\varpi_1} = \frac{dR}{dr}\frac{dr}{d\varpi_1} + \frac{dR}{d\theta}\frac{d\theta}{d\varpi_1}.$$

Also
$$\frac{d\theta}{d\varepsilon_1} + \frac{d\theta}{d\varpi_1} = 1,$$
$$\frac{dr}{d\varepsilon_1} + \frac{dr}{d\varpi_1} = 0;$$

$$\text{therefore} \quad \frac{dR}{d\varepsilon_1} + \frac{dR}{d\varpi_1} = \frac{dR}{d\theta};$$

$$\text{therefore} \quad \frac{dh_1}{dt} = \frac{dR}{d\varepsilon_1} + \frac{dR}{d\varpi_1},$$

$$\text{and} \quad h_1^2 = \mu a_1 (1 - e_1^2);$$

therefore, differentiating the logarithms,

$$\frac{2dh_1}{h_1 dt} = \frac{da_1}{a_1 dt} - \frac{2e_1 \dfrac{de_1}{dt}}{1 - e_1^2};$$

$$\text{therefore} \quad \frac{de_1}{dt} = -\frac{1 - e_1^2}{e_1 h_1}\frac{dh_1}{dt} + \frac{1 - e_1^2}{2e_1 a_1}\frac{da_1}{dt},$$

$$\text{and} \quad \mu = n_1^2 a_1^3;$$

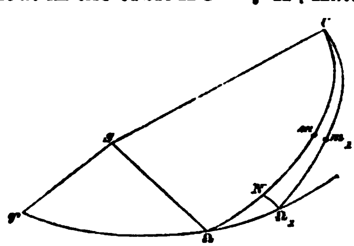
therefore

$$\frac{de_1}{dt} = -\frac{n_1 a_1 \sqrt{(1 - e_1^2)}}{\mu e_1}\left(\frac{dR}{d\varepsilon_1} + \frac{dR}{d\varpi_1}\right) + \frac{n_1 a_1 (1 - e_1^2)}{\mu e_1}\frac{dR}{d\varepsilon_1},$$

which is the rate of change of the eccentricity.

LEMMA I. *To find the change of the longitude of the epoch and perihelion due to a change of the position of the line of nodes.*

13. The longitude of a point m in the orbit $\Omega U = \Upsilon\,\Omega + \Omega m$. If now the plane of the orbit be changed, so as to arrive at the position $Um_1\Omega_1$, m changes to m_1, such that $Um = Um_1$, and longitude of



$$m_1 = \Upsilon\,\Omega_1 + \Omega_1 m_1.$$

Draw $\Omega_1 N$ perpendicular to $UN\Omega$. The increase of the longitude of m equals $\Omega\Omega_1 - \Omega N = \Delta\Omega_1 (1 - \cos i_1)$, ultimately; therefore, if $\Delta\varepsilon_1$ and $\Delta\varpi_1$ be the changes of ε_1 and ϖ_1 due to the change $\Delta\Omega_1$

$$\frac{\Delta\varepsilon_1}{\Delta\Omega_1} = \frac{\Delta\varpi_1}{\Delta\Omega_1} = 1 - \cos i_1, \text{ ultimately.}$$

LEMMA II. *To find the values of* $\dfrac{dr}{de_1}$ *and* $\dfrac{d\theta}{de_1}$, r *and* θ *being expanded in terms of the mean anomaly.*

14. Since r may be expressed in terms of the mean anomaly by eliminating u from the equations

$$r = a_1(1 - e_1 \cos u),$$

$$n_1 t + \epsilon_1 - \varpi_1 = u - e_1 \sin u;$$

therefore
$$\frac{dr}{de_1} = a_1 e_1 \sin u \frac{du}{de_1} - a_1 \cos u,$$

$$0 = (1 - e_1 \cos u)\frac{du}{de_1} - \sin u;$$

therefore
$$\frac{1}{a_1}\frac{dr}{de_1} = \frac{e_1 \sin^2 u}{1 - e_1 \cos u} - \cos u$$

$$= \frac{e_1 - \cos u}{1 - e_1 \cos u} = \frac{1}{e_1} \cdot \frac{1 - e_1 \cos u - (1 - e_1^2)}{1 - e_1 \cos u}$$

$$= \frac{1}{e_1}\left\{1 - \frac{a_1(1 - e_1^2)}{r}\right\} = -\cos(\theta - \varpi_1);$$

therefore
$$\frac{dr}{de_1} = -a_1 \cos(\theta - \varpi_1).$$

Similarly, θ may be obtained from the equation

$$\tan\frac{\theta - \varpi_1}{2} = \sqrt{\left(\frac{1 + e_1}{1 - e_1}\right)} \tan\frac{u}{2},$$

or $\log\{\tan\frac{1}{2}(\theta - \varpi_1)\} = \frac{1}{2}\log(1 + e_1) - \frac{1}{2}\log(1 - e_1) + \log(\tan\frac{1}{2}u)$;

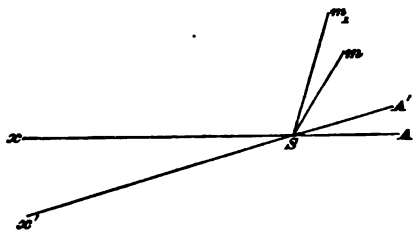
therefore $\dfrac{1}{\sin(\theta - \varpi_1)}\dfrac{d\theta}{de_1} = \dfrac{1}{2}\left(\dfrac{1}{1 + e_1} + \dfrac{1}{1 - e_1}\right) + \dfrac{1}{\sin u}\dfrac{du}{de_1}$;

therefore
$$\frac{d\theta}{de_1} = \left(\frac{1}{1 - e_1^2} + \frac{a_1}{r}\right)\sin(\theta - \varpi_1).$$

To calculate the rate of change of the longitude of perihelion of the instantaneous ellipse.

15. When a body is moving in an undisturbed elliptic orbit, the velocity is the resultant of two constant velocities, viz. $\frac{\mu e}{h}$ perpendicular to the major axis, and $\frac{\mu}{h}$ perpendicular to the radius vector.

Let ASx, $A'Sx'$ be the positions of the major axes of the instantaneous ellipses at the times t and $t+\delta t$, m, m_1, the positions of the planet at those times, $ASA'=\delta\varpi_1$, $mSm_1=\delta\theta$, supposing the motion of the plane of the instantaneous orbit not to be taken into account.



At the time $t+\delta t$, the velocity of m is the resultant of two velocities $\frac{\mu e_1}{h_1}+\delta\frac{\mu e_1}{h_1}$ perpendicular to $A'Sx'$ and $\frac{\mu}{h_1}+\delta\frac{\mu}{h_1}$ perpendicular to Sm_1. Hence, the velocity of m_1 estimated parallel to $A'Sx'=\left(\frac{\mu}{h_1}+\delta\frac{\mu}{h_1}\right)\sin A'Sm_1$.

But this velocity is the component of the velocity in that direction in the orbit constructed for the time t, corresponding to the vectorial angle $\theta+\delta\theta-\varpi_1$, together with the velocity due to the action of the disturbing force in that direction which is $\frac{dR}{dx'}$; therefore

$$\left(\frac{\mu}{h_1}+\delta\frac{\mu}{h_1}\right)\sin(m_1SA')=\frac{\mu}{h_1}\sin(m_1SA')-\frac{e_1\mu}{h_1}\sin\delta\varpi_1+\frac{dR}{dx'}\delta t;$$

therefore, neglecting terms of the second order of small quantities depending upon δt,

$$\delta\frac{\mu}{h_1}.\sin(\theta-\varpi_1)+\frac{e_1\mu}{h_1}d\varpi_1-\frac{dR}{dx}\delta t=0;$$

therefore $$\frac{e_1\mu}{h_1}\frac{\delta\varpi_1}{\delta t}=\frac{dR}{dx}+\frac{\mu}{h_1^2}\frac{dR}{d\theta}\sin(\theta-\varpi_1).$$

But resolving the forces in the direction Sx,

$$\frac{dR}{dx}=\frac{dR}{dr}\{-\cos(\theta-\varpi_1)\}+\frac{dR}{rd\theta}\sin(\theta-\varpi_1);$$

therefore

$$\frac{e_1\mu}{h_1}\frac{\delta\varpi_1}{\delta t}=-\frac{dR}{dr}\cos(\theta-\varpi_1)+\frac{dR}{d\theta}\left\{\frac{1}{r}+\frac{1}{a_1(1-e_1^2)}\right\}\sin(\theta-\varpi_1)$$

$$=\frac{1}{a_1}\left\{\frac{dR}{dr}.\frac{dr}{de_1}+\frac{dR}{d\theta}\frac{d\theta}{de_1}\right\}=\frac{1}{a_1}\frac{dR}{de_1};$$

therefore $$\frac{\delta\varpi_1}{\delta t}=\sqrt{\left(\frac{1-e_1^2}{a_1\mu e_1^2}\right)}\frac{dR}{de_1},$$

which is ultimately the rate of change required, as far as the motion of the perihelion on the orbit is concerned. It remains to determine the term which must be added in consequence of the motion of the plane of the orbit itself.

Now the change of the longitude of the perihelion which is due to the change of the position of the line of nodes is $(1-\cos i_1)\,\Delta\Omega_1$;

therefore $$\frac{d\varpi_1}{dt} = \text{limit}\,\frac{\delta\varpi_1 + (1-\cos i_1)\,\delta\Omega_1}{\delta t};$$

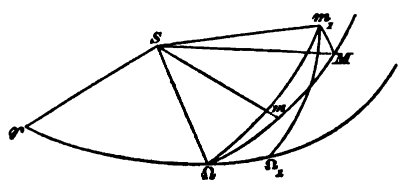
therefore the rate of change, taking this motion into account,

$$= \frac{d\varpi_1}{dt} = \sqrt{\left(\frac{1-e_1^2}{a_1\mu e_1^2}\right)}\,\frac{dR}{de_1} + (1-\cos i_1)\,\frac{d\Omega}{dt}$$

$$= \frac{n_1 a_1\sqrt{(1-e_1^2)}}{\mu e_1}\,\frac{dR}{de_1} + \frac{n_1 a_1 \tan\frac{1}{2} i_1}{\mu\sqrt{(1-e_1^2)}}\cdot\frac{dR}{di_1}\ \text{(See Art. 16).}$$

To calculate the rate of change of the longitude of the node of the plane of the instantaneous ellipse.

16. Let Ω_1, $\Omega_1 + \delta\Omega_1$ be the longitudes of the node at the times t, $t+\delta t$, θ the longitude of the planet at the time t measured as usual on the plane of the orbit Ωm found for the time t; m_1 the position of the planet at the time $t+\delta t$, SM the projection of Sm_1 on the plane Ωm.



Let $m_1SM = \phi$, z' the distance from the plane.

The velocity at the time $t+\delta t$ perpendicular to the plane $m_1SM = \dfrac{rd\theta}{dt} = \dfrac{h_1}{r'}$ ultimately, and the velocity perpendicular to the plane $\Omega m = \dfrac{dR}{dz'}\,\delta t$. If therefore χ be the inclination of the plane of the instantaneous orbit at the time $t+\delta t$ to Ωm,

$$\tan\chi = \frac{r}{h_1}\,\frac{dR}{dz'}\,\delta t.$$

Let a sphere cut the directions of Sm, SM, &c.

Then $$\frac{\sin\delta\Omega_1}{\sin(\theta-\Omega_1)} = \frac{\sin\chi}{\sin i_1}$$

i_1 being the inclination to the fixed plane $\Upsilon\Omega$ of reference; therefore, neglecting quantities of the second order in δt,

$$\delta\Omega = \frac{\sin(\theta-\Omega_1)}{\sin i_1} \tan\chi = \frac{r \sin(\theta-\Omega_1)}{h_1 \sin i_1} \frac{dR}{dz'} \delta t.$$

Now, if we consider Ω_1 constant while i_1 alone varies, a displacement $\Delta z'$ may be considered as due to a change Δi_1 of i_1 alone, in which case $m_1\Omega M = \Delta i_1$, and from the right-angled triangle $m_1\Omega M$,

$$\tan\phi = \sin(\theta-\Omega_1)\tan\Delta i_1$$

$$\text{and } \Delta z' = r\tan\phi \text{ ultimately,}$$

$$= r\sin(\theta-\Omega_1)\,\Delta i_1;$$

therefore proceeding to the limit

$$\frac{dR}{dz'} = \frac{1}{r\sin(\theta-\Omega_1)}\frac{dR}{di_1},$$

(Ω_1 being considered constant in the differentiation);

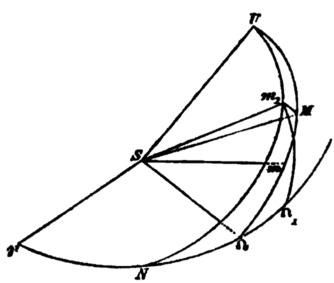
therefore
$$\frac{d\Omega_1}{dt} = \frac{1}{h_1 \sin i_1}\frac{dR}{di_1}$$

$$= \frac{n_1 a_1}{\mu\sqrt{1-e_1^2}\sin i_1}\frac{dR}{di_1},$$

which is the rate of change of the longitude of the node.

To calculate the rate of change of the inclination of the plane of the instantaneous orbit.

17. Let i_1, $i_1+\delta i_1$ be the inclinations to the fixed plane of



reference of the planes of the instantaneous orbits constructed for the times t and $t+\delta t$.

Then employing the same method as before,

$$\cos(i_1+\delta i_1) = \cos\chi\cos i_1 - \sin\chi\sin i_1\cos(\theta-\Omega_1);$$

therefore, neglecting terms of a higher order than the first,

$$-\sin i_1 \delta i_1 = -\chi \sin i_1 \cos(\theta - \Omega_1);$$

therefore $$\delta i_1 = \frac{r\cos(\theta-\Omega_1)}{h_1}\frac{dR}{dz'}\delta t.$$

Consider now i_1 constant, while Ω_1 only varies, a displacement $\Delta z'$ may be considered as due to a change $\Delta\Omega_1$ of Ω_1 alone, which is the same thing as letting the plane ΩM revolve about SU perpendicular to $S\Omega$ to the position Um_1N.

For, by the quadrantal triangle $NU\Omega$,

$$\cos UN\Omega = \cos i \cos NU\Omega;$$

therefore $UN\Omega = i$ to the first order, or the inclination is unaltered, and by the right-angled triangle m_1UM,

$$\tan m_1M = \tan(m_1UM)\cos(\theta - \Omega_1),$$

$$\text{and } \sin(m_1UM) = -\Delta\Omega_1 \sin i_1, \text{ since } N\Omega_1 = -\Delta\Omega_1;$$

therefore $\Delta z' = r\tan m_1M = -\Delta\Omega_1 . r\cos(\theta-\Omega_1)\sin i_1$, ultimately;

therefore $$\frac{dR}{dz'} = -\frac{1}{r\cos(\theta-\Omega_1)\sin i_1} . \text{limit}\,\frac{\Delta R}{\Delta\Omega_1},$$

where ΔR denotes the change of R due to the change of Ω_1 whether as involved explicitly, or implicitly in ε_1 and ϖ_1;

therefore $$\Delta R = \frac{dR}{d\Omega_1}\Delta\Omega_1 + \frac{dR}{d\varepsilon_1}\Delta\varepsilon_1 + \frac{dR}{d\varpi_1} . \Delta\varpi_1,$$

ε_1 and ϖ_1 being the only elements affected by a change in Ω_1,

and $$\frac{\Delta\varepsilon_1}{\Delta\Omega_1} = 1 - \cos i_1 = \frac{\Delta\varpi_1}{\Delta\Omega_1};$$

therefore $$\frac{di_1}{dt} = -\frac{1}{h_1\sin i}\left\{\frac{dR}{d\Omega_1} + \left(\frac{dR}{d\varepsilon_1} + \frac{dR}{d\varpi_1}\right)(1-\cos i_1)\right\}$$

$$= -\frac{n_1a_1}{\mu\sqrt{(1-e_1^2)}}\left\{\frac{1}{\sin i_1}\frac{dR}{d\Omega_1} + \tan\frac{i_1}{2}\left(\frac{dR}{d\varepsilon_1} + \frac{dR}{d\varpi_1}\right)\right\},$$

which is the rate of change required.

NOTE. That the plane Ωm may revolve about a line SU perpendicular to $S\Omega$, so as to change the position of the line of nodes without altering the inclination, is obvious from the fact that this effect is produced by making the plane revolve about a normal to ♈$S\Omega$, to which it may be supposed rigidly attached; and this is equivalent to two rotations, one about a normal to itself, and the other about SU, and the former produces no alteration in either the position of the line of nodes, or the inclination.

ON A GENERALISATION GIVEN BY LAPLACE OF LAGRANGE'S THEOREM.

By Dr. SCHLAEFLI, Professor of Mathematics at the University of Bern.

1. THE theorem which we are about to explain may be enunciated as follows:

"Let $F(x_1, x_2, \dots x_n)$ denote any given function of the n "variables $x_1, x_2, \dots x_n$, which by means of the n equations

$$x_m = t_m + \alpha_m \phi_m (x_1, x_2, \dots x_n), \quad [m = 1, 2, 3, \dots n] \dots (1),$$

"depend on the $2n$ variables $t_1, t_2, \dots t_n, \alpha_1, \alpha_2, \dots \alpha_n$, consider "these as the independent variables and assume that the "functions $F, \phi_1, \phi_2, \dots \phi_n$, in their explicit form, contain no "other variables than the dependent ones $x_1, \dots x_n$. Then "$a_1, a_2, \dots a_n$ being positive integers, we have

$$\frac{d^{a_1+a_2+\dots+a_n} F}{d\alpha_1^{a_1} d\alpha_2^{a_2} \dots d\alpha_n^{a_n}} = \frac{d^{a_1+a_2+\dots+a_n-n}}{dt_1^{a_1-1} dt_2^{a_2-1} \dots dt_n^{a_n-1}} \left[\frac{d^n F}{d\alpha_1 d\alpha_2 \dots d\alpha_n}\right],$$

"where the brackets may indicate that, after having trans-"formed the included derivative into a rational and integral "expression comprising only derivations with regard to "$t_1, t_2, \dots t_n$, the quantities $\phi_1, \phi_2, \dots \phi_n$ are to be replaced "by $\phi_1^{a_1}, \phi_2^{a_2}, \dots \phi_n^{a_n}$."

Laplace gives this theorem (see *Méc. cél.* t. I. p. 175, of the first edition) for the purpose of expanding $F(x_1, x_2, \dots x_n)$ in ascending powers and products of the n independent variables $\alpha_1, \alpha_2, \dots \alpha_n$; and therefore it is only for the case of their vanishing that he wants to know the corresponding derivatives of F, which can then be explicitly expressed in terms of $t_1, t_2, \dots t_n$, and require no more regard to the implicit connexion between all the variables of the system. Accordingly, Laplace's proof of this theorem holds only with the restriction that $\alpha_1, \alpha_2, \dots \alpha_n$ all vanish. But the theorem is generally true; and that is what I presently endeavour to show, though I have not succeeded in effecting the proof in so simple a manner as Laplace does for the special case mentioned.

2. To show briefly that $\dfrac{d^n F}{d\alpha_1 d\alpha_2 \dots d\alpha_n}$ can be expanded into a series of terms composed of $\phi_1, \phi_2, \dots \phi_n$ and the derivatives of these functions and of F, all taken with respect

to $t_1, t_2, \dots t_n$ alone, we may content ourselves with the first steps of the successive transformation. In the first equation $x_1 = t_1 + \alpha_1\phi_1$ of the system (1) and in F, if we conceive the dependent variables $x_2, x_3, \dots x_n$, by means of the remaining equations, to be expressed in terms of $t_2, t_3, \dots t_n$, $\alpha_2, \alpha_3, \dots \alpha_n$ and x_1, we shall easily find that

$$\frac{dF}{d\alpha_1} = \phi_1 \frac{dF}{dt_1}.$$

This result, depending in no manner upon the substitutions made use of, but solely on $t_1, t_2, \dots t_n$, $\alpha_1, \alpha_2, \dots \alpha_n$ being regarded as the independent variables, may be extended to every function of $x_1, x_2, \dots x_n$ alone and to each index. We then have

$$\frac{d^2F}{d\alpha_1\, d\alpha_2} = \frac{d\phi_1}{d\alpha_2}\frac{dF}{dt_1} + \phi_1 \frac{d}{dt_1}\left(\frac{dF}{d\alpha_2}\right) = \phi_2 \frac{d\phi_1}{dt_2}\frac{dF}{dt_1} + \phi_1 \frac{d}{dt_1}\left(\phi_2 \frac{dF}{dt_2}\right)$$

$$= \phi_2 \frac{d\phi_1}{dt_2}\frac{dF}{dt_1} + \phi_1 \frac{d\phi_2}{dt_1}\frac{dF}{dt_2} + \phi_1\phi_2 \frac{d^2F}{dt_1\, dt_2},$$

and so on. Now let the symbol $\left(\begin{matrix} F \\ \phi_1, \phi_2, \dots \phi_n \end{matrix}\right)$ denote the expansion of $\dfrac{d^nF}{d\alpha_1\, d\alpha_2 \dots d\alpha_n}$ just explained, and put $\left(\begin{matrix} F \\ \phi_1^{a_1}, \phi_2^{a_2}, \dots \phi_n^{a_n} \end{matrix}\right)$ instead of it, when therein throughout $\phi_1, \phi_2, \dots \phi_n$ have been replaced by $\phi_1^{a_1}, \phi_2^{a_2}, \dots \phi_n^{a_n}$. The theorem to be proved may then be expressed as follows:

$$\frac{d^{a_1+a_2+\dots+a_n} F}{d\alpha_1^{a_1}\, d\alpha_2^{a_2} \dots d\alpha_n^{a_n}} = \frac{d^{a_1+a_2+\dots+a_n-n}}{dt_1^{a_1-1}\, dt_2^{a_2-1} \dots dt_n^{a_n-1}} \left(\begin{matrix} F \\ \phi_1^{a_1}, \phi_2^{a_2}, \dots \phi_n^{a_n} \end{matrix}\right) \dots (2).$$

3. The proof depends chiefly on the combinatory character of the new symbolical expression and can only be effected under the supposition that all the combinatory formulæ we are concerned with are already admitted as true, for a system of less than n equations. As the given system (1) will be the one and only subject of our consideration, always keeping its n equations, it may be proper to add a remark tending to illustrate how we can nevertheless speak of less than n equations. The expansion of a derivative of F with regard to $\alpha_1, \alpha_2, \dots \alpha_m$ (where $m < n$), for instance, has the very same form as for a system of only m equations with $2m$ independent variables; for it is obvious that in the m first equations of the system (1), by the help of the $n - m$ remain-

ing ones, we may substitute the values of the dependent variables $x_{m+1}, x_{m+2}, \ldots x_n$ in terms of $t_{m+1}, t_{m+2}, \ldots t_n, \alpha_{m+1}, \alpha_{m+2} \ldots \alpha_n$ and $x_1, x_2, \ldots x_m$, and that then the same rules apply as to a genuine system of m equations.

Whenever one or more pairs of corresponding variables t, α are not be permuted with the remaining pairs, we shall, for distinctness, write them by other letters, such as u, β, or v, γ, and the corresponding functions by χ, ψ, and the exponents by b, c. Again, sets of powers of the several ϕ's in the symbols may be shortly indicated at by Φ, Φ', Φ'', ..., with the meaning that the sets Φ, Φ', ..., employed in a single product of such symbols, shall always exhaust the whole of those of the powers ϕ^a which are admitted into permutation. After this preparation, the combinatory character of the symbol in question is defined by the equation

$$\binom{F}{\chi^b, \phi_2^{a_2}, \phi_3^{a_3}, \ldots \phi_n^{a_n}} = \Sigma \binom{\chi^b}{\Phi} \frac{d}{du} \binom{F}{\Phi'} \ldots\ldots (3),$$

where the sum extends to all partitions into two sets Φ, Φ' of the powers $\phi_2^{a_2}, \phi_3^{a_3}, \ldots \phi_n^{a_n}$, inclusively of those partitions where all the powers fall within one set, and the other set disappears. If Φ', for instance, denote the set which disappears, then $\binom{F}{\Phi'}$ will be $= F$; and if the set Φ at first disappears and at last comprehends all the powers ϕ^a, then the sum in (3) begins with

$$\chi^b \frac{d}{du} \binom{F}{\phi_2^{a_2}, \phi_3^{a_3}, \ldots \phi_n^{a_n}},$$

and ends with

$$\binom{\chi^b}{\phi_2^{a_2}, \phi_3^{a_3}, \ldots \phi_n^{a_n}} \frac{dF}{du}.$$

Now, it may first be shown that the symbolical expression defined by (3) has in fact a symmetrical form. By writing v, γ, ψ, c instead of t_2, α_2, ϕ_2, a_2 the above equation becomes changed into

$$\binom{F}{\chi^b, \psi^c, \Phi} = \Sigma \binom{\chi^b}{\Phi'} \frac{d}{du} \binom{F}{\psi^c, \Phi''} + \Sigma \binom{\chi^b}{\psi^c, \Phi'} \frac{d}{du} \binom{F}{\Phi''},$$

where each sum extends to all partitions of the given set Φ, that is $\phi_3^{a_3}, \phi_4^{a_4}, \ldots \phi_n^{a_n}$, into two sets Φ' and Φ''. But since the symbols on the right-hand refer at most to $n-1$

equations, we assume for them the formula (3) as granted; we thus obtain

$$\Sigma \begin{pmatrix} \chi^b \\ \Phi' \end{pmatrix} \frac{d}{du} \begin{pmatrix} F \\ \psi^c, \Phi'' \end{pmatrix} = \Sigma \begin{pmatrix} \chi^b \\ \Phi' \end{pmatrix} \begin{pmatrix} \psi^c \\ \Phi'' \end{pmatrix} \frac{d^2}{du\,dv} \begin{pmatrix} F \\ \Phi''' \end{pmatrix}$$

$$+ \Sigma \begin{pmatrix} \chi^b \\ \Phi' \end{pmatrix} \cdot \frac{d}{du} \begin{pmatrix} \psi^c \\ \Phi'' \end{pmatrix} \cdot \frac{d}{dv} \begin{pmatrix} F \\ \Phi''' \end{pmatrix},$$

$$\Sigma \begin{pmatrix} \chi^b \\ \psi^c, \Phi' \end{pmatrix} \frac{d}{du} \begin{pmatrix} F \\ \Phi'' \end{pmatrix} = \Sigma \begin{pmatrix} \psi^c \\ \Phi'' \end{pmatrix} \cdot \frac{d}{dv} \begin{pmatrix} \chi^b \\ \Phi' \end{pmatrix} \cdot \frac{d}{du} \begin{pmatrix} F \\ \Phi''' \end{pmatrix},$$

and by adding the two expressions and collecting under one sign of derivation the terms which can be so collected, and applying again the formula (3),

$$\begin{pmatrix} F \\ \chi^b, \psi^c, \Phi \end{pmatrix} = \Sigma \begin{pmatrix} \psi^c \\ \Phi'' \end{pmatrix} \frac{d}{dv} \left\{ \begin{pmatrix} \chi^b \\ \Phi' \end{pmatrix} \frac{d}{du} \begin{pmatrix} F \\ \Phi''' \end{pmatrix} \right\}$$

$$+ \Sigma \begin{pmatrix} \chi^b \\ \Phi' \end{pmatrix} \frac{d}{du} \begin{pmatrix} \psi^c \\ \Phi'' \end{pmatrix} \times \frac{d}{dv} \begin{pmatrix} F \\ \Phi''' \end{pmatrix}$$

$$= \Sigma \begin{pmatrix} \psi^c \\ \Phi'' \end{pmatrix} \frac{d}{dv} \begin{pmatrix} F \\ \chi^b, \Phi' \end{pmatrix} + \Sigma \begin{pmatrix} \psi^c \\ \chi^b, \Phi'' \end{pmatrix} \frac{d}{dv} \begin{pmatrix} F \\ \Phi' \end{pmatrix};$$

that is to say, that an exchange of $t_1, \phi_1^{a_1}$ for $t_2, \phi_2^{a_2}$ does not affect the symbolical expression; and as the same reason holds for every other index than 2, the asserted symmetry in fact exists.

It is proper to remark that $\begin{pmatrix} F \\ \Phi \end{pmatrix}$, as containing in general only derivatives of F with respect to the variables t, must vanish whenever F is a constant, and that it reduces itself then only to this constant F, when the set Φ disappears. Consequently, if, for instance, the exponent $a_1 = b$ is equal to zero, the formula (3) becomes

$$\begin{pmatrix} F \\ \chi^0, \Phi \end{pmatrix} = \frac{d}{du} \begin{pmatrix} F \\ \Phi \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

4. *Theorem.* If a function, containing explicitly no variables but the dependent ones x, be a product FG, we shall have the equation

$$\begin{pmatrix} FG \\ \phi_1^{a_1}, \phi_2^{a_2}, \ldots \phi_n^{a_n} \end{pmatrix} = \Sigma \begin{pmatrix} F \\ \Phi \end{pmatrix} \begin{pmatrix} G \\ \Phi' \end{pmatrix} \quad \ldots\ldots\ldots (5).$$

Dem. Admit this formula as true for systems of less than n equations. Then, putting u, χ, b instead of t_1, ϕ_1, a_1, by the help of (3) and (5) we shall obtain

$$\left(\frac{FG}{\chi^b, \phi_2^{a_2}, \phi_3^{a_3}, \dots \phi_n^{a_n}}\right) = \Sigma \left(\frac{\chi^b}{\Phi}\right) \frac{d}{du} \left(\frac{FG}{\Phi'}\right)$$

$$= \Sigma \left(\frac{\chi^b}{\Phi}\right) \left(\frac{F}{\Phi'}\right) \frac{d}{du} \left(\frac{G}{\Phi''}\right) + \Sigma \left(\frac{\chi^b}{\Phi}\right) \left(\frac{G}{\Phi''}\right) \frac{d}{du} \left(\frac{F}{\Phi'}\right)$$

$$= \Sigma \left(\frac{F}{\Phi}\right) \left(\frac{G}{\chi^b, \Phi'}\right) + \Sigma \left(\frac{F}{\chi^b, \Phi}\right) \left(\frac{G}{\Phi'}\right).$$

If, therefore, the formula (5) holds for less than n equations, it must continue to hold for a system of n equations. Now $\left(\frac{F}{\phi^a}\right)$ is $= \phi^a \frac{dF}{dt}$; consequently

$$\left(\frac{FG}{\phi^a}\right) = \phi^a F \frac{dG}{dt} + \phi^a G \frac{dF}{dt} = F \left(\frac{G}{\phi^a}\right) + \left(\frac{F}{\phi^a}\right) G,$$

that is, the formula (5) is true for a single equation, and therefore it is generally so. Hence and from (3) follows the corollary

$$\left(\frac{F}{\chi^{b+b'}, \phi_2^{a_2}, \phi_3^{a_3}, \dots \phi_n^{a_n}}\right) = \Sigma \left(\frac{\chi^b}{\Phi}\right) \left(\frac{\chi^{b'}}{\Phi'}\right) \frac{d}{du} \left(\frac{F}{\Phi''}\right)$$

$$= \Sigma \left(\frac{\chi^{b'}}{\phi}\right) \left(\frac{F}{\chi^b, \phi'}\right) \dots\dots\dots (6).$$

5. The *principal subsidiary theorem*, upon which the proof of the formula (2) depends, may thus be expressed:

$$\frac{d}{d\beta} \left(\frac{F}{\chi^b, \phi_2^{a_2}, \phi_3^{a_3}, \dots \phi_n^{a_n}}\right) = \frac{d}{du} \left(\frac{F}{\chi^{b+1}, \phi_2^{a_2}, \phi_3^{a_3}, \dots \phi_n^{a_n}}\right) \dots (7);$$

but, when $b = 0$, it is to be written thus:

$$\frac{d}{d\beta} \left(\frac{F}{\phi_2^{a_2}, \phi_3^{a_3}, \dots \phi_n^{a_n}}\right) = \left(\frac{F}{\chi, \phi_2^{a_2}, \phi_3^{a_3}, \dots \phi_n^{a_n}}\right) \dots (8).$$

Dem. We have, in virtue of (3),

$$\frac{d}{d\beta} \left(\frac{F}{\chi^b, \Phi}\right) = \Sigma \frac{d}{d\beta} \left(\frac{\chi^b}{\Phi'}\right) \cdot \frac{d}{du} \left(\frac{F}{\Phi''}\right) + \Sigma \left(\frac{\chi^b}{\Phi'}\right) \frac{d^2}{du\, d\beta} \left(\frac{F}{\Phi''}\right).$$

But, if the truth of (8) be granted for less than n equations, we here shall have

$$\frac{d}{d\beta} \left(\frac{F}{\Phi}\right) = \left(\frac{F}{\chi, \Phi}\right) = \Sigma \left(\frac{\chi}{\Phi'}\right) \frac{d}{du} \left(\frac{F}{\Phi''}\right),$$

and likewise when F is replaced by χ^b. Thus we obtain

$$\frac{d}{d\beta}\begin{pmatrix}F\\ \chi^b,\ \Phi\end{pmatrix} = \Sigma\begin{pmatrix}\chi\\ \Phi'\end{pmatrix}\cdot\frac{d}{du}\begin{pmatrix}\chi^b\\ \Phi''\end{pmatrix}\cdot\frac{d}{du}\begin{pmatrix}F\\ \Phi'''\end{pmatrix}$$

$$+\Sigma\begin{pmatrix}\chi^b\\ \Phi''\end{pmatrix}\cdot\frac{d}{du}\left\{\begin{pmatrix}\chi\\ \Phi'\end{pmatrix}\frac{d}{du}\begin{pmatrix}F\\ \Phi'''\end{pmatrix}\right\}$$

$$=\frac{d}{du}\Sigma\begin{pmatrix}\chi^b\\ \Phi''\end{pmatrix}\begin{pmatrix}\chi\\ \Phi'\end{pmatrix}\frac{d}{du}\begin{pmatrix}F\\ \Phi'''\end{pmatrix},$$

and, according to (6),

$$\frac{d}{d\beta}\begin{pmatrix}F\\ \chi^b,\ \Phi\end{pmatrix} = \frac{d}{du}\begin{pmatrix}F\\ \chi^{b+1},\ \Phi\end{pmatrix}.$$

This is the general formula (7) for n equations, and it still remains to prove the particular formula (8). We have

$$\frac{d}{d\beta}\begin{pmatrix}F\\ \psi^c,\ \phi_3^{a_3},\ \phi_4^{a_4},\ \dots\ \phi_n^{a_n}\end{pmatrix} = \frac{d}{d\beta}\Sigma\begin{pmatrix}\psi^c\\ \Phi\end{pmatrix}\frac{d}{dv}\begin{pmatrix}F\\ \Phi'\end{pmatrix}$$

$$=\Sigma\begin{pmatrix}\psi^b\\ \Phi\end{pmatrix}\frac{d^2}{dv\,d\beta}\begin{pmatrix}F\\ \Phi'\end{pmatrix}+\Sigma\frac{d}{d\beta}\begin{pmatrix}\psi^c\\ \Phi\end{pmatrix}\cdot\frac{d}{dv}\begin{pmatrix}F\\ \Phi'\end{pmatrix},$$

and, by employing the formula (8) itself, if granted for less than n equations,

$$=\Sigma\begin{pmatrix}\psi^c\\ \Phi\end{pmatrix}\frac{d}{dv}\begin{pmatrix}F\\ \chi,\ \Phi'\end{pmatrix}+\Sigma\begin{pmatrix}\psi^c\\ \chi,\ \Phi\end{pmatrix}\frac{d}{dv}\begin{pmatrix}F\\ \Phi'\end{pmatrix}$$

$$=\begin{pmatrix}F\\ \psi^c,\ \chi,\ \phi_3^{a_3},\ \phi_4^{a_4},\ \dots\ \phi_n^{a_n}\end{pmatrix}.$$

Lastly, if we consider that for one equation $x = t + a\phi$ the formula (8) $\frac{dF}{da} = \phi\,\frac{dF}{dt} = \begin{pmatrix}F\\ \phi\end{pmatrix}$ is true, we shall find that the proof of the general theorem is now complete.

6. If the symbolical expression in (3) be only expanded in respect to the several derivatives of F, it will assume the following form:

$$\begin{pmatrix}F\\ \phi_1^{a_1},\ \phi_2^{a_2},\dots\phi_n^{a_n}\end{pmatrix}=\Sigma\begin{pmatrix}\phi_{m+1}^{\ a_{m+1}}\ \phi_{m+2}^{\ a_{m+2}},\ \dots\times\phi_n^{\ a_n}\\ \phi_1^{a_1},\ \phi_2^{a_2},\ \dots\ \phi_m^{a_m}\end{pmatrix}\frac{d^{n-m}F}{dt_{m+1}\,dt_{m+2}\dots dt_n}$$

............(9).

The sum here extends to all the possible partitions into two sets, of all the indices 1, 2, 3, ... n. Though the first set (here 1, 2, 3, ... m) employed in the lower row of the symbolical expression on the right-hand may disappear, yet the second set (here $m+1,\ m+2, \dots n$) employed in the product at the upper place of the symbol, must contain at least one index. The first term of the sum is therefore

$$\phi_1^{a_1}\,\phi_2^{a_2} \dots \phi_n^{a_n}\,\frac{d^n F}{dt_1\,dt_2\,\dots\,dt_n},$$

and the row of the n last terms begins with

$$\left(\begin{matrix} \phi_1^{a_1} \\ \phi_2^{a_2},\ \phi_3^{a_3}, \dots \phi_n^{a_n} \end{matrix}\right)\frac{dF}{dt_1};$$

the number of all the terms is of course $2^n - 1$.

The proof of (9), if granted for an inferior system, is easy, but irksome to be written; it is only based upon the formulæ (3) and (5).

7. If we want again to expand the expression (9) in respect to the functions $\phi_1, \phi_2, \dots \phi_n$ and their derivatives, so that there shall be no more symbols but the usual ones of differentiation, then we must in every possible way distribute all the indices $1, 2, 3, \dots n$ into any number of sets (of course at least one set, at most n sets). Within one and the same set, it matters not how the indices may be arranged; but each permutation of the same sets (one set with another) is to be counted as a distinct arrangement. For any such arrangement of the indices, take the products of those among the given powers $\phi_1^{a_1}, \phi_2^{a_2}, \dots \phi_n^{a_n}$, which correspond to the single sets of indices, and put them in the same order in a horizontal line. Then take likewise the products of the symbols of derivation $\frac{d}{dt_1}, \frac{d}{dt_2}, \dots \frac{d}{dt_n}$, corresponding to the same sets of indices in the same order as before, and prefix the compounded symbols of operation thus obtained to the above products of powers, so that the first remains unoperated on, but that each of the following products of powers is operated on by the immediately foregoing symbol, and that the last symbol operates on F. Now multiply all these results. Then the aggregate of all such products will be the complete expansion of the expression (9).

By way of illustration let us assume $n = 10$, and as one of the many possible partitions of the indices, take

$$3, 4;\ \ 2, 5, 6;\ \ 1, 10;\ \ 7, 8, 9.$$

To this arrangement then will correspond the following term of the expansion, viz.

$$\phi_3^{\alpha_3}\phi_4^{\alpha_4}.\frac{d^2(\phi_2^{\alpha_2}\phi_5^{\alpha_5}\phi_6^{\alpha_6})}{dt_3\,dt_4}.\frac{d^3(\phi_1^{\alpha_1}\phi_{10}^{\alpha_{10}})}{dt_2\,dt_5\,dt_6}.\frac{d^2(\phi_7^{\alpha_7}\phi_8^{\alpha_8}\phi_9^{\alpha_9})}{dt_1\,dt_{10}}.\frac{d^3F}{dt_7\,dt_8\,dt_9}.$$

The number of terms in such complete expansion of the expression (9) is equal to the coefficient of $x^n \div 1.2.3\ldots n$ in the expansion of $1 \div (2 - e^x)$ developed in ascending powers of x. If we denote it by A_n, it may be calculated from the recurrent formula

$$A_n = \sum_{\lambda=0}^{\lambda=n-1} \frac{n(n-1)\ldots(n-\lambda+1)}{1.2\ldots\lambda} A_\lambda,$$

which gives us $A_0 = 1$, $A_1 = 1$, $A_2 = 3$, $A_3 = 13$, $A_4 = 75$, $A_5 = 541$, etc.

Bern, *Jan.* 1856.

ON THE A POSTERIORI DEMONSTRATION OF THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

By A. Cayley.

In my former paper "On the Porism of the In-and-circumscribed Triangle" (*Journal*, t. I. p. 344) the two porisms (the homographic and the allographic) were established *à priori*, i.e. by means of an investigation of the order of the curve enveloped by the third side of the triangle. I propose in the present paper to give the *à posteriori* demonstration of these two porisms; first according to Poncelet and then in a form not involving (as do his demonstrations) the principle of projections. My objection to the employment of the principle may be stated as follows: viz. that in a systematic development of the subject, the theorems relating to a particular case and which are by the principle in question extended to the general case, are not in anywise more simple or easier to demonstrate than are the theorems for the general case; and, consequently that the circuity of the method can and ought to be avoided.

The porism (homographic) of the in-and-circumscribed triangle, viz.—

If a triangle be inscribed in a conic, and two of the sides envelope conics having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

The following is Poncelet's demonstration, the numbers are those of the *Traitè des Proprietès Projectives:*

No. 431. If a triangle be inscribed in a circle and two of the sides are parallel to given lines, then the third side envelopes a concentric circle.

This is evident, for the angle in the segment subtended by the third side being constant, the length of the third side is constant; hence, the length of the perpendicular from the centre upon the third side is also constant and the third side envelopes a concentric circle.

Hence, by the principle of projections—

If a triangle be inscribed in a conic and two of the sides pass through given points, the remaining side envelopes a conic having double contact with the circumscribed conic, the line through the two points being the chord of contact.

No. 434. Conversely, if there be a triangle inscribed in a conic and the first side envelope a conic having double contact with the circumscribed conic, and the second side pass through a fixed point in the chord of contact, then will the third side also pass through a fixed point in the chord of contact.

No. 437. In particular, if there be a triangle inscribed in a conic and two of the sides pass through fixed points, then will the third side pass through a fixed point, viz. the point forming with the other two points a conjugate system.

No. 439. It follows that—

If there be a triangle inscribed in a conic and the first side passes through a fixed point, and the second side envelopes a conic having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

For the chord of contact meets the polar of the fixed point with respect to the circumscribed conic in a point; the line joining this point with the third angle (i.e. the angle opposite the third side) of the triangle meets the conic in a variable point; and joining this variable point with the first and second angles of the triangle we have a new triangle; two of the sides of this new triangle (by Nos. 434 and 437) pass through fixed points; hence the remaining side, i.e. the third side of the original triangle, touches a conic having double contact with the circumscribed conic.

We have thus passed from the case of the two sides passing through fixed points to that of one of the two sides enveloping a conic having double contact with the given conic and the other of them passing through a fixed point; and, by a repetition of the reasoning, Poncelet passes to the general case, viz.

If there be a triangle inscribed in a conic, and two of the sides envelope conics having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

But it is somewhat more simple to omit the intermediate case of a conic and point, and pass directly, by the reasoning of No. 439., from the case of two points to that of two conics.

In fact, considering the point of intersection of the two chords of contact, the line joining this point with the third angle of the triangle meets the conic in a variable point, and joining this variable point with the first and second angles of the triangle we have a new triangle: two of the sides of this new triangle (by No. 434.) pass through fixed points; hence the remaining side, i.e. the third side of the original triangle, envelopes a conic having double contact with the circumscribed conic; and the general case is thus established.

The porism (allographic) of the in-and-circumscribed triangle, viz.—

If a triangle be inscribed in a conic and two of the sides envelope conics meeting the circumscribed conic in the same four points, then the third side will touch a conic meeting the circumscribed conic in the four points.

The following is Poncelet's demonstration:

No. 433. In the particular case of the homographic porism, viz.—that in which two of the sides of the triangle pass through fixed points and the remaining side envelopes a conic having double contact with the circumscribed conic—it is easy to see that the lines joining the angles of the triangle with the two fixed points and with the point of contact on the third side, meet in a point; this follows at once by the principle of projection from the case in No. 431., viz. the case of a triangle inscribed in a circle when two of the sides are parallel to given lines and the third side touches a concentric circle. Hence,

No. 531. If there be a triangle inscribed in a conic, and two of the sides envelope fixed curves, and the third side envelopes a certain curve; the lines joining the angles of the triangle with the points of contact meet in a point.

In fact, attending only to the infinitesimal variation of the position of the triangle, the curves enveloped by the first and second sides may be replaced by the points of contact on these sides, and the curve enveloped by the third side may be replaced by a conic having double contact with the circumscribed conic, and the general case thus follows at once from the particular one.

Nos. 162 and 163. LEMMA.* If, on the sides of a triangle ABC, there are taken any three points L, M, N in the same line, and the harmonics A', B', C' of these points (i.e. the harmonic of each point with respect to the two vertices on the same side of the triangle), then the lines AA', BB', CC' will meet in a point; and, moreover, if $A'L$, $B'M$, $C'N$ are bisected in F, G, H (or, what is the same thing, if $\overline{FA'}^2 = FB.FC$, $\overline{GB'}^2 = GC.GA$, $\overline{HC'}^2 = HA.HB$), then will the three points F, G, H lie in a line. This is, in fact, the theorem No. 164.—In any complete quadrilateral the middle points of the three diagonals lie in a line.

It is now easy to prove a particular case of the allographic porism, viz.

No. 531. If there be a triangle inscribed in a circle, such that two of the sides envelope circles having a common secant (real or ideal) with the circumscribed circle; then will the third side envelope a circle having the same common secant with the circumscribed circle.

For if the triangle be ABC, and the points of contact of the sides CB, CA with the enveloped circles and the point of contact of the side AB with the enveloped curve, be A', B', C'; if moreover the points of intersection of the circumscribed circle and the two enveloped circles be M, N, and the common secant MN meet the sides of the triangle in F, G, H; then F, G, H and A', B', C' are points on the sides of the triangle ABC, such that F, G, H lie in a line, and AA', BB', CC' meet in a point. And by a property of the circle

$$\overline{FA'}^2 = FM.FN = FB.FC,$$

$$\overline{GB'}^2 = GM.GN = GC.GA.$$

Whence by the lemma (or rather its converse) $\overline{HC'}^2 = HA.HB$ and by a property of the circle $HA.HB = HM.HN$; and therefore, $\overline{HC'}^2 = HM.HN$, a property which can only belong to a circle having, with the other circles, the common secant MN: the particular case is thus demonstrated. And the

* I have not thought it necessary to give the figures; they can be supplied without difficulty.

principle of projections leads at once to the general case of the allographic porism.

To exhibit the demonstrations in a form independent of the principle of projections, it will be convenient to enunciate the following three lemmas: the first of them being, in fact, the theorem contained in No. 434, as generalised by No. 531; the second of them a theorem connected with and including the properties of the circle assumed in Poncelet's demonstration of the allographic porism; and the third of them a theorem derivable by the principle of projections from the theorem in Nos. 162 and 163.

LEMMA I. If there be a triangle inscribed in a conic, such that two of the sides envelope given curves and the third side envelopes a curve; then the lines joining the angles of the triangle with the points of contact of the opposite sides meet in a point.

LEMMA II. If there be three conics meeting in the same four points, then any line meets the conic in six points forming a system in involution.

COROLL. 1. If the line be a tangent to one of the conics, then the point of contact is the double or sibi-conjugate point of the involution formed by the points of intersection with the other two conics. And conversely if the curve enveloped by the line is not given, but the preceding property holds for all positions of the tangent line; then the curve enveloped by such line is a conic passing through the points of intersection of the two given conics.

COROLL. 2. If one of the conics be a pair of coincident lines, then the other two conics are conics having double contact, with the line in question for their chord of contact; any line meets the chord of contact in a point which is a double or sibi-conjugate point of the involution formed by the points of intersection with the other two conics; and if the line be a tangent to one of the conics, then the point of contact and the point of intersection with the chord of contact are harmonics with respect to the points of intersection with the other conic. And conversely if every tangent of a curve intersect a line and conic in such manner that the point of contact and the point of intersection with the line are harmonics with respect to the points of intersection with the conic; then the curve is a conic having double contact with the given conic, and the line in question is the chord of contact.

The third lemma is a theorem (first explicitly stated, so far as I am aware, by Steiner, Lehrsätze 24 and 25, Crelle, t. XIII. p. 212, and demonstrated by Bauer, t. XIX. p. 227) which, in a note in the *Phil. Mag.*, Augt. 1853, I have called the *Theorem of the harmonic relation of two lines with respect to a quadrilateral.*

LEMMA III. If on each of the diagonals of a quadrilateral there be taken two points harmonically related with respect to the angles upon this diagonal; then if three of the points lie in a line, the other three points will also lie in a line: the two lines are said to be harmonically related with respect to the quadrilateral.

The relation may be exhibited under a different form. The three diagonals of the quadrilateral form a triangle, the sides of which contain the six angles of the quadrilateral; and considering only three of the six angles (one angle on each diagonal) these three angles are points which either lie in a line, or else are such that the lines joining them with the opposite angles of the triangle meet in a point. Each of the three points is, with respect to the involution formed by the two angles of the triangle and the two points harmonically related thereto, a double or sibi-conjugate point, and we have thus a theorem of the harmonic relation of two lines to a triangle and line, or else to a triangle and point, viz. Theorem. If on the sides of a *triangle* there be taken three points which either lie in a *line* or else are such that the lines joining them with the opposite angles of the triangle meet in a *point;* and if on each side of the triangle there be taken two points forming with the two angles on the same side an involution having the first-mentioned point on the same side for a double or sibi-conjugate point; then if three of the six points lie in a *line*, the other three of the six points will also lie in a line; the two lines are said to be harmonically related to the triangle and line, or (as the case may be) to the triangle and point.

The proof of the two porisms is by the preceding lemmas rendered very simple.

Demonstration of the homographic porism.

First, the particular case, where two of the sides pass through fixed points. Lemma I. gives the construction of the point of contact on the third side, and the figure shews that the point of contact and the point in which the third side is intersected by the line through the two given points are harmonics with respect to the points of intersection of the third side with the circumscribed conic. Hence, (Lemma II. Coroll. 2.) the curve touched by the third side is a conic having double contact with the circumscribed conic, and the chord of contact is the line joining the two given points; and conversely if one of the sides touch a conic having double contact with the circumscribed conic and another of the sides passes through a fixed point on the chord of contact, then the third side will also pass through a fixed point on the chord of contact. The general case is deduced from the particular one precisely as before, viz. where two of the sides touch conics having double contact with the circumscribed conic, then considering the point of intersection of the two chords of contact, the line joining this point with the third angle of the triangle meets the circumscribed conic in a variable point, and joining this variable point with the first and second angles of the triangle, we have a new triangle two of the sides of which (by the converse of the particular case) pass through fixed points: hence the remaining side, i.e. the third side of the original triangle, touches a conic having double contact with the circumscribed conic.

Demonstration of the allographic porism.

Let ABC be the triangle, A', B', C' the points of contact on the three sides, then by Lemma I. the lines AA', BB', CC' meet in a point. Take a pair of lines passing through the points of intersection of the circumscribed conic with the two given conics enveloped by the sides CA, CB, and let one of these lines meet the sides of the triangle in the points F, G, H, and the other of them meet the sides of the triangle in the points F', G', H'. Then considering the following three conics, viz. the last-mentioned pair of lines, the circumscribed conic, and the conic enveloped by the side CA; these are conics passing through the same four points, and the side CA is a tangent to one of them; hence by Lemma II. Coroll. 1., G, G', C, A will be an involution having the point B' for a double or sibi-conjugate point, and similarly F, F', G, B are an involution having the point A' for a double or sibi-conjugate point. It follows from Lemma III. that H, H', A, B will be an involution having C' for a double or sibi-conjugate point. Hence by Lemma II. Coroll. 1. (the two given conics being the before-mentioned pair of lines and the circumscribed conic) the curve enveloped by the side AB will be a conic passing through the points of intersection of the pair of lines and the circumscribed conic, or, what is the same thing, the points of intersection of the circumscribed conic and the conics enveloped by the other two sides.

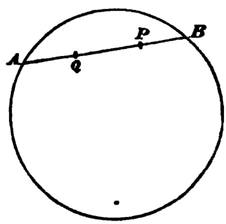
2, Stone Buildings, *Oct.* 2, 1856.

GEOMETRICAL THEOREM.

By Rev. Hamnet Holditch.

If a chord of a closed curve, of constant length $c+c'$, be divided into two parts of lengths c, c' respectively, the difference between the areas of the closed curve, and of the locus of the dividing point, will be $\pi cc'$.

Solution. Let AB be the chord in any position; P the dividing point, so that $AP=c$, $BP=c'$; let Q be the point in which the chord intersects its consecutive position; let $[A]$ be the area of the given curve, $[P]$, $[Q]$, those of the loci of P, Q, respectively; $AQ=r$, $BQ=c+c'-r$.



Then $$[A]-[Q]=\tfrac{1}{2}\int_0^{2\pi} r^2 d\theta \quad\text{.................. (1)},$$

but also $$[A]-[Q]=\tfrac{1}{2}\int_0^{2\pi} (c+c'-r)^2\, d\theta;$$

therefore $$\tfrac{1}{2}\int_0^{2\pi} r^2 d\theta=\tfrac{1}{2}\int_0^{2\pi} (c+c'-r)^2\, d\theta,$$

or $$(c+c')\int_0^{2\pi} r\,d\theta=\tfrac{1}{2}\int_0^{2\pi} (c+c')^2\, d\theta;$$

therefore $$\int_0^{2\pi} r\,d\theta=\pi(c+c') \quad\text{.................. (2)}.$$

Also $$[P]-[Q]=\tfrac{1}{2}\int_0^{2\pi} (c-r)^2\, d\theta,$$

therefore, by (1), $$[A]-[P]=\tfrac{1}{2}\int_0^{2\pi} (2cr-c^2)\, d\theta$$
$$=c\int_0^{2\pi} r\,d\theta-\pi c^2$$
$$=\pi c(c+c')-\pi c^2, \text{ by (2)},$$
$$=\pi cc'.$$

NOTE ON THE SYMMETRICAL EXPRESSION OF THE CONSTANTS IN THE EQUATIONS OF ALGEBRAICAL CURVES.

By Samuel Roberts, M.A.

If the right line $x\cos\alpha' + y\sin\alpha' - p' = 0$ be thrown into the form $lL + mM + nN = 0$ (Salmon's *Conics*, p. 55), where

$$L = x\cos\alpha_1 + y\sin\alpha_1 - p_1,$$

$$M = x\cos\alpha_2 + y\sin\alpha_2 - p_2,$$

$$N = x\cos\alpha_3 + y\sin\alpha_3 - p_3,$$

the coefficient l must be of the form

$$l = \frac{\sin A}{Det.(\cos\alpha', \sin\alpha', -p')}\,a_1;$$

A being the angle between $M = 0$ and $N = 0$, and a_1 being the perpendicular from the intersection of those lines on the line

$$a = x\cos\alpha' + y\sin\alpha' - p' = 0 \ldots\ldots\ldots\ldots(a).$$

Similarly,

$$m = \frac{\sin B}{Det.(\cos\alpha', \sin\alpha', -p')}\,a_2,$$

where B is the angle between $L = 0$ and $N = 0$, and a_2 is the perpendicular from LN on (a).

In like manner

$$n = \frac{\sin C}{Det.(\cos\alpha', \sin\alpha', -p')}\,a_3,$$

where C is the angle between $L = 0$ and $M = 0$, and a_3 is the perpendicular from LM on (a).

Consequently, if we treat in the same manner the lines

$$\beta = x\cos\alpha'' + y\sin\alpha'' - p'' = 0,$$

$$\gamma = x\cos\alpha''' + y\sin\alpha''' - p''' = 0,$$

and $l_1, m_1, n_1, l_2, m_2, n_2$ be the corresponding coefficients of the transformed equations, we shall have (l, l_1, l_2), (m, m_1, m_2), (n, n_1, n_2) proportional, respectively to the trilinear coordinates of MN, LN, LM, with regard to the lines α, β, γ.

If then we transform the equation $\phi(x', y', z') = 0$ into

$$\phi(a_1x + b_1y + c_1,\ a_2x + b_2y + c_2,\ a_3x + b_3y + c_3) = 0,$$

the quantities $(a_1a_2a_3)$, $(b_1b_2b_3)$, $(c_1c_2c_3)$ will respectively be the trilinear coordinates of (yz), (xz), (xy) with respect to x', y', z'. By the aid of Taylor's theorem the transformed equation is easily developed. Thus, using the notation

$$\underset{ab}{\Delta} = a_1\frac{dU_b}{db_1} + a_2\frac{dU_b}{db_2} + a_3\frac{dU_b}{db_3},$$

where $\qquad U_b = \phi(b_1b_2b_3)$,

one of the numerous symmetrical forms, which the development assumes, will be

$$\left.\begin{aligned}
& U_a x^n + U_b y^n + U_c z^n \\
+ & \underset{ab}{\Delta} xy^{n-1} + \underset{ba}{\Delta} yx^{n-1} + \underset{ac}{\Delta} xz^{n-1} + \underset{ca}{\Delta} zx^{n-1} + \underset{bc}{\Delta} yz^{n-1} + \underset{cb}{\Delta} zy^{n-1} \\
+ & \frac{1}{1.2}\{\underset{ab}{\Delta^2} x^2y^{n-2} + \underset{ba}{\Delta^2} y^2x^{n-2} + \underset{ac}{\Delta^2} x^2z^{n-2} + \underset{ca}{\Delta^2} z^2x^{n-2} + \underset{bc}{\Delta} y^2z^{n-2} + \underset{cb}{\Delta} z^2y^{n-2}\} \\
+ & \underset{ab}{\Delta}\underset{cb}{\Delta} xzy^{n-2} + \underset{ac}{\Delta}\underset{bc}{\Delta} xyz^{n-2} + \underset{ca}{\Delta}\underset{ba}{\Delta} yzx^{n-2} + \dots\dots\dots
\end{aligned}\right\}$$

$$\dots\dots\dots\dots(b),$$

Δ being a symbol of operation and the corresponding subjects U_a, U_b, U_c being suppressed. It is immediately seen that the form of the development may be varied by the use of the identity

$$\frac{1}{1.2\dots k}\underset{mn}{\Delta^k} U = \frac{1}{1.2\dots n-k}.\underset{nm}{\Delta^{n-k}} U,$$

and a corresponding identity for $\underset{ab}{\Delta^k}.\underset{cb}{\Delta^{n-k}}$.

The development (b) shews that if LM, MN, NL are on the curve, the terms containing a variable in the n^{th} degree vanish; for

$$U_a = U_b = U_c = 0.$$

And if each of the lines of reference is the polar of the intersection of the other two, the terms containing a variable in the $n-1^{\text{th}}$ degree vanish, and so on.

It evidently enables us to treat the general equations of curves expressed in trilinear coordinates in a manner analogous to that in which their Cartesian equations are usually dealt with, namely, by transformation of coordinates; and although many of the conclusions thus derived can quite as easily be deduced from other considerations, the above method appears less chargeable with artifice.

Applied to the general equation of the second degree, the transformation above given becomes

$$U_a x^2 + U_b y^2 + U_c z^2 + \underset{ab}{\Delta} xy + \underset{ac}{\Delta} xz + \underset{bc}{\Delta} yz = 0 \ldots (1).$$

Take $U_a = U_b = U_c = 0$, and we have

$$\underset{ab}{\Delta} xy + \underset{ac}{\Delta} xz + \underset{bc}{\Delta} yz = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

representing conics circumscribing the triangle of reference. Take $U_b = U_c = 0$, $\underset{ab}{\Delta} = \underset{ac}{\Delta} = 0$, and we have

$$U_a x^2 + \underset{bc}{\Delta} yz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Take $\underset{ab}{\Delta} = \underset{ac}{\Delta} = \underset{bc}{\Delta} = 0$, and we have

$$U_a x^2 + U_b y^2 + U_c z^2 = 0 \ldots\ldots\ldots\ldots\ldots (4),$$

the triangle of reference being self-conjugate.

In the case of conics, we obviously have

$$\underset{ab}{\Delta} = \underset{ba}{\Delta} = \sin A \sin B . R' p_{ba} = \sin A \sin B . R'' p_{ab},$$

$$\underset{ac}{\Delta} = \underset{ca}{\Delta} = \sin A \sin C . R''' p_{ac} = \sin A \sin C . R' p_{ca},$$

$$\underset{bc}{\Delta} = \underset{cb}{\Delta} = \sin B \sin C . R''' p_{bc} = \sin B \sin C . R'' p_{cb},$$

$\underset{kl}{\Delta}$ now meaning the result of the operation

$$k_1 \frac{dU_l}{dl_1} + k_2 \frac{dU_l}{dl_2} + k_3 \frac{dU_l}{dl_3},$$

ABC being as before the angles of the transformed triangle of reference, and p_{kl} denoting the perpendicular from $(k_1 k_2 k_3)$ to the polar of $(l_1 l_2 l_3)$.

Again, for U_a we may put $\dfrac{\underset{aa}{\Delta}}{2}$ or $\dfrac{\sin^2 A . R' p_{aa}}{2}$,

for U_b .... $\dfrac{\underset{bb}{\Delta}}{2}$ or $\dfrac{\sin^2 B . R'' p_{bb}}{2}$,

for U_c .... $\dfrac{\underset{cc}{\Delta}}{2}$ or $\dfrac{\sin^2 C . R''' p_{cc}}{2}$.

And since $\underset{ab}{\Delta}^2 = \underset{ab}{\Delta}.\underset{ba}{\Delta} = \sin^2 A \sin^2 B . R'R'' p_{ab} p_{ba}.$

And there are similar values for $\underset{ac}{\Delta}^2$ and $\underset{bc}{\Delta}^2$, we may write (1) in the form

$$\sin^2 A R' p_{aa} x^2 + \sin^2 B R'' p_{bb} y^2 + \sin^2 C R''' p_{cc} z^2$$
$$+ 2 \{\sin A \sin B \sqrt{(R'R'' p_{ab} p_{ba})}\, xy + \sin A \sin C \sqrt{(R'R''' p_{ac} p_{ca})}$$
$$+ \sin B \sin C \sqrt{(R''R''' p_{bc} p_{cb})}\, yz\} = 0.$$

But if the polars of A and B intersect on the line AB, we shall have

$$p_{aa} . p_{bb} = p_{ab} . p_{ba}.$$

Therefore this is a condition that AB may touch the conic, and substituting this and two similar conditions for AC, BC, we have the general equation of a conic inscribed in the triangle of reference.

The general equation of a circle inscribed in the triangle is readily obtained, for we have

$$p_{aa} = r_1 \cos\frac{A}{2}, \quad p_{bb} = r_2 \cos\frac{B}{2}, \quad p_{cc} = r_3 \cos\frac{C}{2},$$

$$p_{ba} = r_2 \cos\frac{A}{2}, \quad p_{ab} = r_1 \cos\frac{B}{2}, \quad p_{ac} = r_1 \cos\frac{C}{2},$$

$$p_{ca} = r_3 \cos\frac{A}{2}, \quad p_{cb} = r_3 \cos\frac{B}{2}, \quad p_{bc} = r_2 \cos\frac{C}{2},$$

$$\frac{r_1 \sin\frac{A}{2}}{\rho} = \cos\frac{A}{2}, \quad \frac{r_2 \sin\frac{B}{2}}{\rho} = \cos\frac{B}{2}, \quad \frac{r_3 \sin\frac{C}{2}}{\rho} = \cos\frac{C}{2},$$

$$\frac{R'}{R'''} = \frac{p_{ac}}{p_{ca}} \quad \frac{R''}{R'''} = \frac{p_{bc}}{p_{cb}}.$$

And substituting and dividing by $\cos\left(\frac{C}{2}\right)$ and ρ^2, we get the general form which may be written

$$x^{\frac{1}{2}} \cos\frac{A}{2} + y^{\frac{1}{2}} \cos\frac{B}{2} + z^{\frac{1}{2}} \cos\frac{C}{2} = 0.$$

Again, (2) may be written

$$\frac{\sin A}{\sin C}\frac{p_{ab}}{p_{cb}}\, xy + \frac{\sin A}{\sin B}\frac{p_{ac}}{p_{bc}}\, xz + yz = 0,$$

and $$\frac{p_{ab}}{p_{cb}} = \frac{AB}{AC}\frac{\sin\theta}{\sin\phi}, \quad \frac{p_{ac}}{p_{bc}} = \frac{AC}{BC}\frac{\sin\theta'}{\sin\phi'},$$

where θ, ϕ are the angles which AB, CB make with the tangent at B, and θ', ϕ' are the angles which AC, BC make with the tangent at C; (2) therefore becomes

$$\frac{\sin\theta}{\sin\phi}xy + \frac{\sin\theta'}{\sin\phi'}xz + yz = 0.$$

Now, for a circle, the angle made by a chord with the tangent at its extremity is equal to the angle in the alternate segment; therefore,

$$\sin Cxy + \sin Bxz + \sin Ayz = 0,$$

is the equation of the circumscribed circle. It appears also that if AB, CB make equal angles with the tangent at B of the conic, and AC, BC make equal angles with the tangent at C, then BA, CA make equal angles with the tangent at A.

(3) may be written

$$\sin^2 A . R'p_{aa}x^2 + 2\sin A\sin B . R''p_{cb}xy = 0,$$

where $R'R''$ are determinate functions of the coordinates of A and C, the angles ABC and the constants of (1), but the ratio $\dfrac{R'}{R''} = \dfrac{p_{ab}}{p_{ba}}$ is illusory.

So (4) may be written

$$\sin^2 A . R'x^2 + \sin^2 B . R'y^2 + \sin^2 C . R'''z^2 = 0.$$

And the like may be said of $R'R''R'''$.

We can express the general equation of the second degree in terms of the angles of the triangle of reference, and the tangential coordinates of their polars in various ways. For (1) may be written

$$\begin{aligned} &R'\sin A\,\{\sin Ap_{aa}x^2 + \sin Bp_{ba}xy + \sin Cp_{ca}xz\} \\ &+ R''\sin B\,\{\sin Bp_{bb}y^2 + \sin Ap_{ab}xy + \sin Cp_{cb}yz\} \\ &+ R'''\sin C\,\{\sin Cp_{cc}z^2 + \sin Ap_{ac}xz + \sin Bp_{bc}yz\} = 0, \end{aligned}$$

where $$\frac{R'}{R'''} = \frac{p_{ac}}{p_{ca}}, \quad \frac{R''}{R'''} = \frac{p_{bc}}{p_{cb}}, \quad \frac{R'}{R''} = \frac{p_{ab}}{p_{ba}},$$

and $$p_{ab} . p_{bc} . p_{ca} = p_{ba} . p_{cb} . p_{ac}.$$

The equation

$$ax^3 + by^3 + cz^3 - 6dxyz = 0,$$

represents the general equation of a cubic referred to three lines containing its nine points of inflexion. It appears also that it represents a cubic referred to a self-conjugate triangle with respect to it. Hence the nine points of inflexion of a cubic lie on each of four self-conjugate triangles, and a cubic has four and no more than four self-conjugate triangles. Hence, also, all cubics having the same nine points of inflexion have the same self-conjugate triangles.

I think the above examples, although elementary, shew that the method may be advantageously employed for the symmetrical and systematic demonstration of theorems commonly proved by isolated transformations; and it would seem to be also applicable in some degree to transcendental curves.

Lincoln, October 1, 1856.

ON CERTAIN FORMS OF THE EQUATION OF A CONIC.

By A. Cayley.

To find the general equation of a conic which passes through two given points and touches a given line.

Let the coordinates of the given points be (α, β, γ), $(\alpha', \beta', \gamma')$ and the equation of the given line be $\lambda x + \mu y + \nu z = 0$. Then writing

$$u = \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix}, \quad v = s\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}, \quad w = \begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix},$$

where a, b, c, s are arbitrary coefficients, the general equation of a conic passing through the two given points will be

$$uw - v^2 = 0.$$

We have identically

$$s\begin{vmatrix} \lambda x + \mu y + \nu z, & x, & y, & z \\ \lambda \alpha + \mu \beta + \nu \gamma, & \alpha, & \beta, & \gamma \\ \lambda \alpha' + \mu \beta' + \nu \gamma', & \alpha', & \beta', & \gamma' \\ \lambda a + \mu b + \nu c, & a, & b, & c \end{vmatrix} = 0;$$

and hence putting

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix}$$

$$A = \quad (\lambda\alpha' + \mu\beta' + \nu\gamma')\, s,$$

$$B = -\,(\lambda a + \mu b + \nu c),$$

$$C = -\,(\lambda\alpha + \mu\beta + \nu\gamma)\, s.$$

We have

$$(\lambda x + \mu y + \nu z)\, s\nabla + Cw + Au + Bv = 0,$$

and consequently the equation $\lambda x + \mu y + \nu z = 0$ is equivalent to

$$Au + Bv + Cw = 0,$$

and we have only to express that the line represented by this equation touches the conic $uw - v^2 = 0$.

Combining the two equations, we find $Au + Cw + B\sqrt{(uw)} = 0$, i.e. $(Au + Cw)^2 - B^2uw = 0$, an equation which must have equal roots; and the condition for this is obviously $4AC - B^2 = 0$. Or putting the condition under the form $-B + 2\sqrt{(AC)} = 0$ and substituting for A, B, C their values, the condition becomes $\{i = \sqrt{(-1)}$ as usual$\}$

$$\lambda a + \mu b + \nu c + 2is\sqrt{\{(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma')\}} = 0.$$

We have consequently

$$s^2 = -\frac{(\lambda a + \mu b + \nu c)^2}{4(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma')},$$

and the equation of the conic is

$$4(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma') \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix} \begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix}$$

$$+ (\lambda a + \mu b + \nu c)^2 \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}^2 = 0.$$

But the equation of the conic may be obtained in a different form as follows: we may first write $B^2v^2 = 4ACuw$, and then substituting for $-Bv$ the value $(\lambda x + \mu y + \nu z)\, s\nabla + Au + Cw$, the equation becomes

$$\{Au + Cw + (\lambda x + \mu y + \nu z)\, s\nabla\}^2 = 4ACuw.$$

or extracting the root of each side and transposing

$$\{\sqrt{(Au)}+\sqrt{(Cw)}\}^2+(\lambda x+\mu y+\nu z)\,s\nabla=0,$$

and thence

$$\sqrt{(Au)}+\sqrt{(Cw)}+i\sqrt{(s\nabla)}\sqrt{(\lambda x+\mu y+\nu z)}=0,$$

or substituting the values of A, C, ∇, u, w, and omitting the common factor $\sqrt{(s)}$ the equation becomes

$$\sqrt{(\lambda\alpha'+\mu\beta'+\nu\gamma')}\sqrt{\left(\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c\end{vmatrix}\right)}+i\sqrt{(\lambda\alpha+\mu\beta+\nu\gamma)}\sqrt{\left(\begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c\end{vmatrix}\right)}$$

$$+\,i\sqrt{(\lambda x+\mu y+\nu z)}\sqrt{\left(\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ a, & b, & c\end{vmatrix}\right)}=0,$$

a form symmetrically related to the three lines

$$\lambda x+\mu y+\nu z=0,$$

$$\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c\end{vmatrix}=0,\qquad \begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c\end{vmatrix}=0.$$

Let it be required to find the conic passing through the two points (α, β, γ), $(\alpha', \beta', \gamma')$, and touching the three lines

$$\lambda_1x+\mu_1y+\nu_1z=0,\quad \lambda_2x+\mu_2y+\nu_2z=0,\quad \lambda_3x+\mu_3y+\nu_3z=0.$$

The constants a, b, c have to be determined in such manner that the equations obtained from the preceding, by writing successively $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$ for (λ, μ, ν) may represent one and the same equation; the three equations so obtained will therefore subsist simultaneously, and we may from the equations in question eliminate a, b, c; the resulting equation

$$\begin{vmatrix} \sqrt{(\lambda_1x+\mu_1y+\nu_1z)}, & \sqrt{(\lambda_2x+\mu_2y+\nu_2z)}, & \sqrt{(\lambda_3x+\mu_3y+\nu_3z)} \\ \sqrt{(\lambda_1\alpha+\mu_1\beta+\nu_1\gamma)}, & \sqrt{(\lambda_2\alpha+\mu_2\beta+\nu_2\gamma)}, & \sqrt{(\lambda_3\alpha+\mu_3\beta+\nu_3\gamma)} \\ \sqrt{(\lambda_1\alpha'+\mu_1\beta'+\nu_1\gamma')}, & \sqrt{(\lambda_2\alpha'+\mu_2\beta'+\nu_2\gamma')}, & \sqrt{(\lambda_3\alpha'+\mu_3\beta'+\nu_3\gamma')}\end{vmatrix}=0$$

is the equation of the conic in question; this is in fact evident from other considerations.

To find the condition in order that a conic passing through the points (α, β, γ), $(\alpha', \beta', \gamma')$ may touch the four lines

$$\lambda_1x+\mu_1y+\nu_1z=0,\quad \lambda_2x+\mu_2y+\nu_2z=0,$$

$$\lambda_3x+\mu_3y+\nu_3z=0,\quad \lambda_4x+\mu_4y+\nu_4z=0.$$

The relation first obtained between a, b, c, s gives four equations from which these quantities may be eliminated, the resulting equation

$$\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1, & \sqrt{\{(\lambda_1\alpha+\mu_1\beta+\nu_1\gamma)(\lambda_1\alpha'+\mu_1\beta'+\nu_1\gamma')\}} \\ \lambda_2, & \mu_2, & \nu_2, & \sqrt{\{(\lambda_2\alpha+\mu_2\beta+\nu_2\gamma)(\lambda_2\alpha'+\mu_2\beta'+\nu_2\gamma')\}} \\ \lambda_3, & \mu_3, & \nu_3, & \sqrt{\{(\lambda_3\alpha+\mu_3\beta+\nu_3\gamma)(\lambda_3\alpha'+\mu_3\beta'+\nu_3\gamma')\}} \\ \lambda_4, & \mu_4, & \nu_4, & \sqrt{\{(\lambda_4\alpha+\mu_4\beta+\nu_4\gamma)(\lambda_4\alpha'+\mu_4\beta'+\nu_4\gamma')\}} \end{vmatrix} = 0$$

is the required relation.

The preceding investigations apply directly to the circle which is a conic passing through two given points. Thus the equation of a circle touching the three lines

$$\begin{aligned} Ax + By + C &= 0, \\ A'x + B'y + C' &= 0, \\ A''x + B''y + C'' &= 0, \end{aligned}$$

is $\begin{vmatrix} \sqrt{(Ax+By+C)}, & \sqrt{(A'x+B'y+C')}, & \sqrt{(A''x+B''y+C'')} \\ \sqrt{(A+Bi)}, & \sqrt{(A'+B'i)}, & \sqrt{(A''+B''i)} \\ \sqrt{(A-Bi)}, & \sqrt{(A'-B'i)}, & \sqrt{(A''+B''i)} \end{vmatrix} = 0.$

Whence also the equation of a circle touching the three lines

$$\begin{aligned} x\cos\alpha + y\sin\alpha - p &= 0, \\ x\cos\beta + y\sin\beta - q &= 0, \\ x\cos\gamma + y\sin\gamma - r &= 0, \end{aligned}$$

is $\sin\tfrac{1}{2}(\beta-\gamma)\sqrt{(x\cos\alpha+y\sin\alpha-p)}+\sin\tfrac{1}{2}(\gamma-\alpha)\sqrt{(x\cos\beta+y\sin\beta-q)}$

$$+\sin\tfrac{1}{2}(\alpha-\beta)\sqrt{(x\cos\gamma+y\sin\gamma-r)}=0.$$

To rationalise the equation, I remark that an equation $\sqrt{(A)}+\sqrt{(B)}+\sqrt{(C)}=0$ gives in general

$$(1, 1, 1, \bar{1}, \bar{1}, \bar{1})(A, B, C)^2 = 0,$$

and that

$(1, 1, 1, \bar{1}, \bar{1}, \bar{1})[2p\sin^2\tfrac{1}{2}(\beta-\gamma),\ 2q\sin^2\tfrac{1}{2}(\gamma-\alpha),\ 2r\sin^2\tfrac{1}{2}(\alpha-\beta)^2]$

or as it may also be written

$(1,1,1,\bar{1},\bar{1},\bar{1})[p\{1-\cos(\beta-\gamma)\},\ q\{1-\cos(\gamma-\alpha)\},\ r\{1-\cos(\alpha-\beta)\}]^2,$

is identically equal to

$$\begin{aligned} &\{p(\sin\beta-\sin\gamma)+q(\sin\gamma-\sin\alpha)+r(\sin\alpha-\sin\beta)\}^2 \\ +\ &\{p(\cos\beta-\cos\gamma)+q(\cos\gamma-\cos\alpha)+r(\cos\alpha-\cos\beta)\}^2 \\ -\ &\{p\sin(\beta-\gamma)+q\sin(\gamma-\alpha)+r\cos(\alpha-\beta)\}^2. \end{aligned}$$

Hence if we replace p, q, r by

$$x\cos\alpha + y\sin\alpha - p, \quad x\cos\beta + y\sin\beta - q, \quad x\cos\gamma + y\sin\beta - r,$$

the last mentioned expression equated to zero will give the equation of the circle, and we obtain

$$\{\nabla x + p(\sin\beta - \sin\gamma) + q(\sin\gamma - \sin\alpha) + r(\sin\alpha - \sin\beta)\}^2$$
$$+ \{\nabla y - p(\cos\beta - \cos\gamma) - q(\cos\gamma - \cos\alpha) - r(\cos\alpha - \cos\beta)\}^2$$
$$- \{p\sin(\beta - \gamma) + q\sin(\gamma - \alpha) + r\sin(\alpha - \beta)\}^2 = 0,$$

where $\nabla = \sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta)$,

and we have thus the equation of the circle in the usual form with the coordinates of the centre and the radius put in evidence.

The condition that there may be a circle touching the four lines

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$
$$A''x + B''y + C'' = 0,$$
$$A'''x + B'''y + C''' = 0,$$

is by the general formula shown to be

$$\begin{vmatrix} A, & B, & C, & \sqrt{(A^2 + B^2)} \\ A', & B', & C', & \sqrt{(A'^2 + B'^2)} \\ A'', & B'', & C'', & \sqrt{(A''^2 + B''^2)} \\ A''', & B''', & C''', & \sqrt{(A'''^2 + B'''^2)} \end{vmatrix} = 0,$$

which is in fact obvious from other considerations.

NOTE ON THE REDUCTION OF AN ELLIPTIC ORBIT TO A FIXED PLANE.

By A. Cayley.

The principal object of the present note is to obtain an expression for the quantity ε_0 which I call the modified mean longitude at epoch, viz. taking as the elements the longitude of the node, inclination and any four elements which determine the motion in the plane of the orbit, then the longitude measured in the fixed plane or reduced longitude will be a function of the form

$$nt + \varepsilon_0 + \text{periodic terms},$$

where ε_0 is a determinate function of the elements, and it is proposed to find the expression of this function. But as the corresponding formulæ relating to the excentricity and longitude of the pericentre are not in general given as part of the theory of elliptic motion, but occur only, so far as I am aware, in works on the lunar theory, I have thought it desirable to include these formulæ and take as the subject of this note the reduction of an elliptic orbit to a fixed plane. Write

a_1, the semiaxis-major,

e_1, the excentricity $(= \sin \kappa_1)$,

ϖ_1, the longitude of pericentre in orbit,

ε_1, the mean longitude in orbit at epoch,

θ, the longitude of node,

ϕ, the inclination $(= \tan^{-1}\gamma)$,

and moreover

n, the mean motion $\left\{= \sqrt{\left(\frac{\sigma}{a_1^3}\right)}\right\}$.

Where by longitude in orbit is to be understood as usual a longitude measured in the fixed plane as far as the node and from the node in the plane of the orbit: the meaning of ε_1 is perhaps more clearly fixed by saying that $\varepsilon_1 - \varpi_1$ denotes the mean anomaly at epoch.

The elements most nearly corresponding to the above, in the orbit reduced to the fixed plane, are

a_0, the modified semiaxis-major,

e_0, the modified excentricity,

ϖ_0, the modified longitude of pericentre,

ε_0, the modified mean longitude at epoch,

θ, the longitude of node,

ϕ, the inclination $(= \tan^{-1}\gamma)$,

and moreover

n, the mean motion $\left\{\text{not equal to } \sqrt{\left(\frac{\sigma}{a_0^3}\right)}\right\}$.

Where θ, ϕ, n are the same as in the actual orbit, but a_0, e_0, ϖ_0, ε_0 are defined as follows: viz. e_0, ϖ_0 are functions of e_1, ϖ_1, θ, ϕ given by the equations

$$\tan(\varpi_0 - \theta) = \sec\phi \tan(\varpi_1 - \theta),$$

$$e_0 = \frac{e_1 \cos(\varpi_1 - \theta)}{\cos(\varpi_0 - \theta)},$$

a_0 is determined by the condition

$$a_0(1 - e_0^2) = a_1(1 - e_1^2).$$

And ε_0 is determined so that the reduced longitude may be equal to

$$nt + \varepsilon_0 + \text{periodic terms}.$$

It is easy to see that considering the orbit and the fixed plane as great circles of the sphere, and projecting the pericentre upon the fixed plane by an arc *perpendicular to the orbit*, then ϖ_0 denotes the longitude of such projection of the pericentre; and e_0 is equal to e_1 into the secant of the projecting arc. In fact we have a right angled spherical triangle, of which the projecting arc in question is the perpendicular, and the hypothenuse and base of which are $\varpi_0 - \theta$ and $\varpi_1 - \theta$ respectively, and the base angle is the inclination ϕ. It is to be remarked that ϖ_0 is *not* the reduced longitude of the pericentre, an expression that would signify the longitude of the projection of the pericentre by an arc perpendicular to the fixed plane; this is the reason why I have throughout used the word *modified* instead of what would at first sight have appeared the natural one, viz. the word *reduced*. The modified semiaxis-major is obviously a semiaxis-major calculated from the latus rectum of the orbit by means of the modified excentricity e_0.

The relations between e_0, ϖ_0, e_1, ϖ_1 may be written

$$\tan(\varpi_0 - \theta) = \sec\phi \tan(\varpi_1 - \theta),$$
$$e_0 = e_1 \sec\phi \sqrt{\{1 - \sin^2\phi \sin^2(\varpi_1 - \theta)\}},$$

or again

$$\tan(\varpi_1 - \theta) = \cos\phi \tan(\varpi_0 - \theta),$$
$$e_1 = e_0 \sqrt{\{1 - \sin^2\phi \sin^2(\varpi_0 - \theta)\}}.$$

Write now

r_1, the radius vector,
v_1, the longitude in orbit,
λ, the latitude ($= \tan^{-1}s$).

And in like manner

r_0, the reduced radius vector,
v_0, the reduced longitude,
λ, the latitude ($= \tan^{-1}s$).

Then $v_1 - \theta$ and $v_0 - \theta$ are the hypothenuse and base of a right angled spherical triangle, the perpendicular being λ and the angle at the base being ϕ. We have

$$\tan\lambda = \tan\phi \sin(v_0 - \theta),$$
$$\sin\lambda = \sin\phi \sin(v_1 - \theta),$$
$$\tan(v_0 - \theta) = \cos\phi \tan(v_1 - \theta),$$
$$\cos(v_0 - \theta) = \sec\lambda \cos(v_1 - \theta).$$

We have for the radius vector

$$\frac{1}{r_1} = \frac{1}{a_1(1 - e_1^2)} \{1 + e_1 \cos(v_1 - \varpi_1)\},$$

and the reduced radius vector is thence found as follows: viz. we have $r_0 = r_1 \cos\lambda$, that is

$$\frac{1}{r_0} = \frac{1}{a_1(1 - e_1^2)} \{\sec\lambda + e_1 \sec\lambda \cos(v_1 - \varpi_1)\}.$$

But $e_1 \sec\lambda \cos(v_1 - \varpi_1)$

$$= e_1 \sec\lambda \cos\{(v_1 - \theta) - (\varpi_1 - \theta)\}$$
$$= e_1 \sec\lambda \cos(v_1 - \theta) \cos(\varpi_1 - \theta) + e_1 \sec\lambda \sin(v_1 - \theta) \sin(\varpi_1 - \theta)$$
$$= e_1 \sec\lambda \cos(v_1 - \theta) \cos(\varpi_1 - \theta) \{1 + \tan(v_1 - \theta) \tan(\varpi_1 - \theta)\}$$
$$= e_0 \cos(v_0 - \theta) \cos(\varpi_0 - \theta) \{1 + \tan(v_0 - \theta) \tan(\varpi_0 - \theta)\}$$
$$= e_0 \cos(v_0 - \theta) \cos(\varpi_0 - \theta) + e_0 \sin(\varpi_0 - \theta) \sin(\varpi_0 - \theta)$$
$$= e_0 \cos(v_0 - \varpi_0),$$

and by the definition of a_0 we have $a_0(1-e_0^2) = a_1(1-e_1^2)$. Whence

$$\frac{1}{r_0} = \frac{1}{a_0(1-e_0^2)}\{\sec\lambda + e_0\cos(v_0 - \varpi_0)\},$$

which, combined with the equation

$$\tan\lambda = \tan\phi\,\sin(v_0 - \theta),$$

determines the position of the body in terms of the modified elements and of the reduced longitude v_0. Introducing into the two equations $s\,(=\tan\lambda)$ and $\gamma\,(=\tan\phi)$ in the place of λ and ϕ, they become

$$\frac{1}{r_0} = \frac{1}{a_0(1-e_0^2)}\{\sqrt{(1+s^2)} + e_0\cos(v_0 - \varpi_0)\},$$

$$s = \gamma\sin(v_0 - \theta),$$

which is the form in which the equations occur in the lunar theory.

Proceeding now to the formulæ which involve the time, it is to be remarked that the true anomaly and the quotient of the radius vector by the semiaxis-major are given functions of the excentricity and the mean anomaly, and calling for a moment the last mentioned quantities e, ξ, I represent the functions in question by

$$\text{elta}\,(e, \xi),\ \text{elqr}\,(e, \xi).$$

Or more simply when the mean anomaly only is attended to by

$$\text{elta}\,\xi,\ \text{elqr}\,\xi.$$

I have found this notation very convenient as a means of dispensing with the introduction of the excentric anomaly.

The reduced longitude is found in terms of the time by means of the equations

$$\tan(v_0 - \theta) = \sec\phi\,\tan(v_1 - \theta),$$

$$v_1 - \varpi_1 = \text{elta}\,(nt + \varepsilon_1 - \varpi_1),$$

the former equation gives, as is well known,

$$v_0 - \theta = v_1 - \theta - \tan^2\tfrac{1}{2}\phi\,\sin(2v_1 - 2\theta) + \tfrac{1}{2}\tan^4\tfrac{1}{2}\sin(4v_1 - 4\theta) - \&c.,$$

(where the successive coefficients are the reciprocals of the natural numbers) we have therefore

$$v_0 = v_1 - \tan^2\tfrac{1}{2}\phi\,\sin\{(2v_1 - 2\varpi_1) + (2\varpi_1 - 2\theta)\} + \&c.,$$

or, as it may be written,

$$v_0 = v_1 - \tan^2\tfrac{1}{2}\phi\,\{\sin(2v_1 - 2\varpi_1)\cos(2\varpi_1 - 2\theta) + \cos(2v_1 - 2\varpi_1)\sin(2\varpi_1 - 2\theta)\}$$
$$+ \tfrac{1}{2}\tan^4\tfrac{1}{2}\phi\,\{\sin(4v_1 - 4\varpi_1)\cos(4\varpi_1 - 4\theta) + \cos(4v_1 - 4\varpi_1)\sin(4\varpi_1 - 4\theta)\}$$
$$- \&c.,$$

and for the present purpose it is only necessary to attend to the non-periodic part of the function on the right-hand side. Now

$$v_1 - \varpi_1 = \text{elta}\,(nt + \varepsilon_1 - \varpi_1),$$

the non-periodic part of which is $nt + \varepsilon_1 - \varpi_1$. And the non-periodic part of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\mu(v_1 - \varpi_1)$ is given by the equation (62) of Hansen's Memoir "Entwickelung des Products" &c. *Abhand. der K. Sächs Gesellschaft zu Leipsig*, t. II. (1853). In fact, Hansen's $\beta = \dfrac{e_1}{1 + \sqrt{(1 - e_1^2)}} = \tan\tfrac{1}{2}\kappa_1$ and the formula gives for the non-periodic parts

$$\cos\mu(v_1 - \varpi_1) = (-)^\mu \tan^\mu\tfrac{1}{2}\kappa_1\,(1 + \mu\cos\kappa_1),$$
$$\sin\mu(v_1 - \varpi_1) = 0.$$

Hence, substituting these values and putting for the non-periodic part of v_0 the assumed value $nt + \varepsilon_0$, we find

$$\varepsilon_0 = \varepsilon_1 - \tan^2\tfrac{1}{2}\phi\,\tan^2\tfrac{1}{2}\kappa_1\,(1 + 2\cos\kappa_1)\sin(2\varpi_1 - 2\theta)$$
$$+ \tfrac{1}{2}\tan^4\tfrac{1}{2}\phi\,\tan^4\tfrac{1}{2}\kappa_1\,(1 + 4\cos\kappa_1)\sin(4\varpi_1 - 2\theta)$$
$$- \&c.$$

The series on the right-hand side may be summed without difficulty, and we obtain

$$\varepsilon_0 = \varepsilon_1 - \tan^{-1}\left\{\frac{\tan^2\tfrac{1}{2}\phi\,\tan^2\tfrac{1}{2}\kappa_1\sin(2\varpi_1 - 2\theta)}{1 + \tan^2\tfrac{1}{2}\phi\,\tan^2\tfrac{1}{2}\kappa_1\cos(2\varpi_1 - 2\theta)}\right\}$$
$$-2\cos\kappa_1\frac{\tan^2\tfrac{1}{2}\phi\,\tan^2\tfrac{1}{2}\kappa_1\sin(2\varpi_1 - 2\theta)}{1 + 2\tan^2\tfrac{1}{2}\phi\tan^2\tfrac{1}{2}\kappa_1\cos(2\varpi_1 - 2\theta) + \tan^4\tfrac{1}{2}\phi\tan^4\tfrac{1}{2}\kappa_1},$$

in which formula the values of $\tan\tfrac{1}{2}\phi$, $\tan\tfrac{1}{2}\kappa_1$ (in terms of γ, e_1) are $\dfrac{\gamma}{1 + \sqrt{(1 + \gamma^2)}}$, $\dfrac{e_1}{1 + \sqrt{(1 - e_1^2)}}$, and that of $\cos\kappa_1$ is $\sqrt{(1 - e_1^2)}$. We have thus the required expression for the modified mean longitude at epoch, and all the modified elements are now expressed in terms of the original elements.

The following investigation leads to a theorem which it is, I think, worth while to notice. We have

$$r_0^2\,\frac{dv_0}{dt} = \sqrt{\{\sigma a_0\,(1-e_0^2)\}}\,\cos\phi,$$

and thence

$$dt = \frac{a_0^{\frac{3}{2}}\,(1-e_0^2)^{\frac{3}{2}}\,dv_0}{\sqrt{(\sigma)}\,\cos\phi\,\{\sec\lambda + e_0\cos(v_0-\varpi_0)\}^2}$$

$$= \frac{a_1^{\frac{3}{2}}\,(1-e_1^2)^{\frac{3}{2}}\,dv_0}{\sqrt{(\sigma)}\,\sqrt{(1+\gamma^2)}\,[\sqrt{\{1+\gamma^2\sin^2(v_0-\varpi_0)\}}+e_0\cos(v_0-\varpi_0)]^2},$$

or as it may be written

$$\frac{dv_0}{[\sqrt{\{1+\gamma^2\sin^2(v_0-\varpi_0)\}}+e_0\cos(v_0-\varpi_0)]^2} = \sqrt{\left(\frac{\sigma}{a_1^3}\right)}\,(1-e_1^2)^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}}dt$$

$$= \quad n\,(1-e_1^2)^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}}dt.$$

But it is easy to see that if the mean longitude $nt+\varepsilon_0$ is expanded in terms of v_0, the relation between these quantities must be of the form $nt+\varepsilon_0 = v_0 +$ periodic terms. It follows that in the preceding equation the non-periodic part of the function which multiplies dv_0 (the expansion being in multiple cosines of v_0) must be equal to $(1-e_1^2)^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}}$. Hence, putting for e_1 its value, we find that the non-periodic part of

$$\frac{1}{[\sqrt{\{1+\gamma^2\sin^2(v_0-\theta)\}}+e_0\cos(v_0-\varpi_0)]^2}$$

expanded in multiple cosines of v_0 is

$$\left[1-e_0^2\left\{1-\frac{\gamma^2}{1+\gamma^2}\sin^2(\varpi_0-\theta)\right\}\right]^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}},$$

a theorem which might, it is probable, be verified without much difficulty.

2, Stone Buildings,
October, 1856.

AN ATTEMPT TO DETERMINE THE TWENTY-SEVEN LINES UPON A SURFACE OF THE THIRD ORDER, AND TO DIVIDE SUCH SURFACES INTO SPECIES IN REFERENCE TO THE REALITY OF THE LINES UPON THE SURFACE.

By Dr. Schläfli, Professor of Mathematics at the University of Bern.
Translated by A. Cayley.

Preliminary remarks. Contrary to the usual practice I would, in the case of a curve, term *singular* those points only at which Taylor's theorem fails for point coordinates, and where in consequence the tangent ceases to be linearly determined; and in like manner term *singular* those tangents for which the point of contact ceases to be linearly determined. Thus a point of inflexion is not a singular point, but the tangent at such point is a singular tangent. According to the same principle, in the case of a surface, I call singular points those only for which the tangent plane ceases to be linearly determined. I say further that a surface is general as regards order when it has no singular points, general as regards class when it has no singular tangent planes. By class I understand the number of tangent planes which pass through an arbitrary line; by singular tangent planes, the tangent planes for which the point of contact ceases to be linearly determined. By *order* of a curve in space, I mean the number of points in which the curve is intersected by an arbitrary plane, by class (as for surfaces) the number of tangent planes (planes containing a tangent of the curve) which pass through an arbitrary point. On account of their reciprocal relation to curves I guard myself from putting *developable surfaces* on a footing with proper curved surfaces, and call them therefore simply *developables* without the addition of the word surface, since they do not, like proper surfaces, arise from the double motion of a plane, but arise from the simple motion of a plane. I call indeed *degree* of a developable the number of points of intersection with an arbitrary line, but *class* the number of generating planes which pass through an arbitrary point. The representation of an algebraical curve in space requires at least two equations, that is, two surfaces passing through the curve. If these surfaces can be chosen so that their complete intersection is merely the curve in question, such curve may be termed a complete-curve (Vollcurve). But when this is not possible, and the complete intersection of any two surfaces passing through the curve consists *always* of such curve accompanied by one or more other curves, the curve in question is termed a partial-curve (Theilcurve).*

Suppose now that $f(w, x, y, z) = 0$ is the homogeneous equation of an algebraical surface of the n^{th} order; w, x, y, z the coordinates of a point P of the surface, which, as the surface originally given, I will call for shortness the *basis*. Moreover let $D = w'\frac{d}{dw} + x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}$ represent a linear derivation symbol, in which the elements w', x', y', z' denote the coordinates of a point in space, which may be designated by the same letter D: the derivation symbol may be called for shortness the symbol of the point to which it relates. The system $f = 0$, $Df = 0$ expresses that the point D is situate in the tangent plane to the surface at the point D. This plane cuts the basis in a curve ($f = 0$, $Df = 0$, $D^n f = 0$) which has the point of contact as a double point; I will call the curve simply the contact section (Berührungschnitt). Since P is an arbitrary point upon the surface, there are in the contact section two disposable elements; when therefore we add the condition that the curve has a second double point, there remains but one disposable element; and if we assume that there are three double points in all, the plane becomes determinate. In other words, to a general (as regards order) algebraical surface of an order higher than the second, there belongs a developable, the generating planes of which touch the surface in two points. Among these generating planes there are found a determinate number of planes touching the surface in three points. The developable may be termed the doubly circumscribed developable,† the planes the triple tangent planes of the surface. The problem which next presents itself is to determine the curve along which the surface is touched by the doubly circumscribed developable.

* The names Vollcurve and Theilcurve belong to Steiner.

† (Note by the Translator). This is the developable which I have called the node-couple developable; and further on, the osculation curve is that frequently called the parabolic curve and which I have termed the spinode curve; the osculating circumscribed developable is what I have termed the spinode developable, and the self-touching double points what I have termed tacnodes. See my paper "On the Singularities of Surfaces," *Cambridge and Dublin Mathematical Journal*, t. VII. p. 166.

*Suppose as before that (w, x, y, z) are the coordinates of a point P of the basis $f=0$ and moreover that (w', x', y', z'), (w'', x'', y'', z'') are the coordinates of two points lying in the corresponding tangent plane, D', D'' their symbols in respect of P, so that $D'f=0$, $D''f=0$. If then ψ, χ, ω are three new variables, and $\psi P+\chi D'+\omega D''$ denotes a point common to the tangent plane and the basis (i.e. if $\psi w+\chi w'+\omega w''$, $\psi x+\chi x'+\omega x''$, &c. are the coordinates of the point in question) then

$$F(\psi, \chi, \omega) = \tfrac{1}{2}\psi^{n-2}(\chi D'+\omega D'')^2 f+\tfrac{1}{6}\psi^{n-3}(\chi D'+\omega D'')^3 f\ldots + \frac{1}{1.2\ldots n}(\chi D'+\omega D'')^n f=0$$

is the equation of the contact section, where ψ, χ, ω are to be considered as the coordinates of a point in a plane; F is a symbol for the polynome on the right-hand side considered as a function of ψ, χ, ω, the coordinates of P, D, D' being treated as constant. If then the curve besides the double point P (at which point $\chi=0$, $\omega=0$) has another double point Q, then putting for shortness $\frac{dF}{d\psi}=F_\psi$, &c., the equations $F_\psi=0$, $F_\chi=0$, $F_\omega=0$ must be satisfied without χ and ω vanishing. This gives an equation between the coordinates in space of the points P, D, D'', and (as might be expected from the nature of the question) finally an equation containing only w, x, y, z, and which combined with the equation $f=0$ represents the required curve of contact of the doubly circumscribed developable. But since by reason of the double point P the resultant of the polynomes F_ψ, F_χ, F_ω vanishes *identically*, the system must be replaced by a system for which this does not happen; to effect this we may proceed as follows:

The functions F_χ, F_ω may be brought under the forms

$$F_\chi = M\chi+N\omega, \quad F_\omega = P\chi+Q\omega,$$

and the equations $F_\chi=0$, $F_\omega=0$ give therefore

$$\Delta = MQ-NP=0$$

and the function Δ for $\chi=0$, $\omega=0$ reduces itself to

$$\{(D'f)^2(D''f)^2-(D'D''f)^2\}\,\psi^{2(n-2)}.$$

* **Remark.** This section contains an attempt to apply Jacobi's method, given in Crelle's Journal, for the determination of the double tangents of a plane curve, to the doubly circumscribed developable of a surface.

Moreover in the development of

$$G = \frac{2}{n-2}\psi F_\psi - \chi F_\chi - \omega F_\omega = \frac{n}{n-2}\psi F_\psi - nF,$$

the lowest term in respect to χ, ω, is

$$-\frac{n}{n-2}\tfrac{1}{8}\psi^{n-3}(\chi D' + \omega D'')^3 f.$$

Considering now the resultant Θ of the system

$$F_\psi = 0, \quad F_\chi = 0, \quad \Delta = 0,$$

this must be in the first place divisible by the resultant K of the system

$$F_\psi = 0, \quad M = 0, \quad N = 0,$$

and in the next place by

$$\Gamma = (D'f)^2 (D''f)^2 - (D'D''f)^2,$$

since $\chi = 0$, $\omega = 0$, $\Gamma = 0$ are also a solution of the system Θ. But since we have identically

$$\frac{2}{n-2} N\psi F_\psi = NG + (N\chi + Q\omega) F_\chi - \Delta\chi\omega,$$

and since for $\Gamma = 0$ and considering χ, ω as indefinitely small quantities of the first order, the polynomes F_χ, Δ are only of the first order, but G is of the third order, Θ must be divisible by Γ^3.* As regards K there is nothing to shew that a higher power than the first enters as a factor into Θ, and a further examination shews that Θ is in fact divisible only by the first power of K.

In relation to ψ, χ, ω we have F_ψ, F_χ each of the degree $n-1$, Δ of the degree $2(n-2)$ and M, N of the degree $n-2$. The coefficient of a term $\psi^\alpha\chi^\beta\omega^\gamma$ in F_ψ is in regard to the coordinates of the points P, D, D' respectively of the degrees $\alpha+1$, β, γ, in F_χ of the degrees α, $\beta+1$, γ, in M of the degrees α, $\beta+2$, γ, in N of the degrees α, $\beta+1$, $\gamma+1$, in P the same, and in Q of the degrees

* (Note by the Translator). I do not quite understand the reasoning: but if we write $F = A\chi^2 + 2B\chi\omega + C\omega^2$ and take Γ the value of $AC - B^2$ corresponding to $\chi = 0$, $\omega = 0$, then when χ, ω are small $d_\psi A$, $d_\psi B$, $d_\psi C$ are proportional to A, B, C, and the system (Θ) may be written $A\chi^2 + 2B\chi\omega + C\omega^2 = 0$, $A\chi + B\omega = 0$, $\Gamma + A_1\chi + B_1\omega = 0$, the last two equations shew that (putting for shortness $AB_1 - A_1B = T$) $T\chi$, $T\omega$ are respectively equal to $-B\Gamma$, $+A\Gamma$, and substituting these values in the first equation, the left-hand side of the resulting equation contains the factor $(AB^2 - 2BAB + CA^2)\Gamma^2$, which is equal to $A(AC - B^2)\Gamma^2$, i.e. the resultant contains the factor Γ^3.

α, β, $\gamma+2$, consequently in Δ of the degrees α, $\beta+2$, $\gamma+2$. Lastly, Γ is in regard to such coordinates of the degrees $2(n-2)$, 2, 2. It follows that in reference to the coordinates of the three points respectively,
Θ is of the degrees

$$2n(n-1)(n-2),\quad 2(n-1)^3+2(n-1)(n-2),\quad 2(n-1)^3,$$

and K of the degrees

$$n(n-2)^2,\quad (n-1)^2(n-2)+2(n-1)(n-2),\quad (n-1)^2(n-2).$$

Whence $\frac{\Theta}{K\Gamma^3}$ is of the degrees

$$(n-2)(n^2-6),\quad n(n-1)^2-6,\quad n(n-1)^2-6;$$

this resultant will be denoted by Ω $\left(\Omega=\frac{\Theta}{K\Gamma^3}\right)$.

If we put

$$\psi=\psi'+\lambda\chi'+\mu\omega',\quad \chi=\alpha\chi'+\beta\omega',\quad \omega=\gamma\chi'+\delta\omega',$$

then in the new system of coordinates (ψ', χ', ω') the fundamental point P is the same as before, and only the two other points D, D' have assumed arbitrary new positions in the tangent plane of the basis at P. The polynome of the equation of the contact section, considered as expressed in terms of ψ', χ', ω' will have the same properties as the before mentioned polynome, it will have therefore a corresponding resultant Ω'; and since x', x'' are respectively replaced by $\lambda x+\alpha x'+\gamma x''$, $\mu x+\beta x'+\delta x''$ and similarly for the other coordinates, Ω' will be in regard to each of the series of constants λ, α, γ and μ, β, δ of the degree $n(n-1)^2-6$. But since $\psi=0$, $\chi=0$, $\omega=0$ is a solution of the new system, which implies $\alpha\delta-\beta\gamma=0$ without besides having the variable solution $\psi'=0$, $\chi'=0$, $\omega'=0$ as a necessary consequence, Ω' must be divisible by a power of $\alpha\delta-\beta\gamma$, in such manner that the quotient may differ from Ω only by a trivial constant (that is a constant independent of α, β, γ, δ, λ, μ), we must therefore have

$$\Omega'=(\alpha\delta-\beta\gamma)^{n(n-1)^2-6}\Omega,$$

since for $\lambda=\mu=\beta=\gamma=0$, $\alpha=\delta=1$, Ω' and Ω must coincide. Suppose now $df=pdw+qdx+rdy+sdz$, and consequently (since the equation $f=0$ is satisfied)

$$pw+qx+ry+sz=0,$$

whence among other relations

$$(pw+qx)(pw+ry)=qrxy-pswz.$$

And writing

$$D' = q\frac{d}{dw} - p\frac{d}{dx}, \quad D'' = s\frac{d}{dy} - r\frac{d}{dz},$$

the points D', D'' will be on the tangent plane. Putting moreover

$$\psi = \psi' + \frac{pr\chi' - qs\omega'}{pw + qx}, \quad \chi = \frac{rx\chi' + sw\omega'}{pw + qx}, \quad \omega = \frac{pz\chi' + yq\omega'}{pw + qx},$$

we have

$$f(w\psi + q\chi,\ x\psi - p\chi,\ y\psi + s\omega,\ z\psi - r\omega)$$
$$= f(w\psi' + r\chi',\ x\psi' - x\omega',\ y\psi' - p\chi',\ z\psi' + q\omega'),$$

and

$$\Omega' = \left(\frac{pw + ry}{pw + qx}\right)^{n(n-1)^2-6}\Omega,$$

as before, under the supposition $f = 0$. But since as well Ω' as Ω are integral functions of $w, x, y, z : p, q, r, s$, viz. in regard to the first set of the degree $(n-2)(n^2-6)$, and in regard to the second set of the degree $2[n(n-1)^2-6]$, it follows that putting for p, q, r, s the values of these quantities considered as derivatives of the polynome f, we must have identically

$$(pw + qx)^{n(n-1)^2-6}\Omega' - (pw + ry)^{n(n-1)^2-6}\Omega = Vf,$$

where V is a rational and integral function of w, x, y, z. There is nothing from which it would appear that the system $f = 0$, $pw + qx = 0$, $pw + ry = 0$, or what is the same thing $pw = -qx = -ry = sz$ represents a curve and not a mere system of discrete points. But since the curve

$$pw + qx = 0, \quad pw + ry = 0$$

lies wholly in the surface $Vf = 0$, and no part of the curve lies in the surface $f = 0$, the curve must lie wholly in the surface $V = 0$, and the form of the identical equation shews that the curve in question enters as an $[n(n-1)^2-6]$-tuple curve of the surface $V = 0$. Now I believe that whenever a complete curve is represented by the equations $k = 0$, $l = 0$, every surface passing through the curve may be represented by an equation $kt + lu = 0$. From such an axiom it follows that, for the present case, we must have identically

$$V = (pw + qx)^{n(n-1)^2-6}T' - (pw + ry)^{n(n-1)^2-6}T,$$

where T, T' are rational and integral functions. And when this is once granted, it follows from known and strictly demonstrated theorems relating to the divisibility of rational functions, that we must have identically

$$\Omega = (pw + qx)^{n(n-1)^2-6} R + Tf,$$

where R is a rational and integral function.

The required curve of contact was at first contained in the system $\Theta = 0$, $f = 0$, then after the separation of extraneous curves in the system $\Omega = 0$, $f = 0$. This last system in virtue of the relation just obtained breaks up into the multiple system $pw + qx = 0$, $f = 0$, and the unique system $R = 0$, $f = 0$. The former on account of its arbitrariness cannot contain the required curve, which must therefore be contained in the latter system. But R being obviously of the degree $(n-2)(n^3 - n^2 + n - 12)$, the degree of the curve of contact is at most $n(n-2)(n^3 - n^2 + n - 12)$. We proceed to shew that the curve is actually of this degree; from which it will follow that it is a complete curve, that is, that a surface $R = 0$ passes through the curve of contact and intersects the basis only in this curve and in no other curve, if at least the axiom relied upon was not deceptive.

Imagine a cone having for its vertex a point D, circumscribed about the surface, and let it be required to find for this cone the degree g, the class k, the number of double sides d, of cuspidal (stationary) sides r, of double tangent planes t, and of stationary tangent planes w. It is clear that it is only necessary to know three of these six numbers in order to determine the others by means of the same three relations which apply to plane curves, viz.

$$n - r = 3(k - g), \quad g(g-1) = k + 2d + 3r, \quad k(k-1) = g + 2t + 3w.$$

(see Steiner's *Memoir* on the subject, Crelle, t. xlvii., and Liouville, t. xviii. p. 309; also Salmon's *Treatise on the Higher Plane Curves*, p. 91). The curve along which the cone touches the surface is defined by the system $f=0$, $Df=0$; the tangent (when Δ denotes the symbol of one of its points) by $\Delta f = 0$, $D\Delta f = 0$. Comparing this with the system $\delta f = 0$, $\delta^2 f = 0$, which determines the two tangents at the double point of the contact section; it is easy to see that the tangent $P\Delta$ of the curve of contact of the surface and circumscribed cone, and the generating line PD of the cone are harmonically related to the two tangents of the contact section at the double point.* Each generating line therefore

* This also follows easily from the more general theorem: If three surfaces touch at the same point, the pairs of tangents of the three contact sections at the point in question form a pencil in involution.

of the cone which coincides with one of the two tangents at the double point of the contact section will be also a tangent to the curve of contact of the surface with the circumscribed cone, and in particular when the point of contact of the tangent plane is a cusp of the contact section, the tangent of the curve of contact of the surface with the circumscribed cone coincides with the cuspidal tangent of the contact section, so long as the generating line of the cone has any other direction whatever. In the former case the cone has a cuspidal (stationary) generating line, in the latter a stationary tangent plane. For the cuspidal or stationary generating line the conditions are $f=0$, $Df=0$, $D^2f=0$, and we have therefore $r=n(n-1)(n-2)$. For a cusp of the contact section of the basis it is necessary that the system $\Delta f=0$, $\Delta^2 f=0$ should have in reference to the elements of Δ two conicident solutions, which may be expressed by the evanescence of ∇f (the Hessian functional determinant or Hessian). Consequently the stationary tangent planes of the cone are given by the system $f=0$, $Df=0$, $\Delta f=0$, and therefore $w=n(n-1)\times 4(n-2)$. The order g of the cone is the class of the section of the basis by a plane through the vertex of the cone, so that $g=n(n-1)$ and the class k of the cone is the class of the basis, that is, $k=n(n-1)^2$. We have already four of the required numbers, more than enough therefore to determine the two others. We find

$$d \doteq \tfrac{1}{2}n(n-1)(n-2)(n-3),$$
$$t = \tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12).$$

I stop to consider this last number t. Since this represents the number of planes passing through a given point D and touching the basis in two distinct points, it is naturally the class of the doubly circumscribed developable of the basis. But the curve of contact is intersected by the polar surface $Df=0$, obviously only in the pairs of points of contact of the planes through D; consequently the number of these points of intersection is $2t$ and the degree of the curve of contact is

$$\frac{2t}{n-1} = n(n-2)(n^3-n^2+n-12),$$

which was the number above obtained as the maximum limit of the degree of the curve. I am indebted to Dr. Steiner for this process for determining the class of the doubly circumscribed developable. The determination of the order of the circumscribed developable appears to me a very interesting problem. If it were solved, as to which I at present know nothing, we should be in a condition to derive, by means of it, the number of the triple tangent planes of the surface, and generally an explanation of all the singularities which a general (as regards order) surface presents in respect to its class.

The order in question would be determined if it could be found, how often, for example, the right line $w=0$, $x=0$ is intersected by a generating line of the developable. If we retain the symbols

$$D' = q\,\frac{d}{dw} - p\,\frac{d}{dx}, \quad D'' = s\,\frac{d}{dy} - r\,\frac{d}{dz},$$

the generating line in question will pass through the points P and D''. For the second double point (besides P) of the contact section we must have $\chi = 0$. The former system, the resultant of which was Ω, then easily reduces itself to the following:

$$\sum_{i=2}^{i=n-1} \frac{n-i}{1.2.3\ldots i}.\,\psi^{n-i-1}\omega^{i-2} \quad D''^{i}f = 0,$$

$$\sum_{i=3}^{i=n} \frac{i-2}{1.2.3\ldots i}.\,\psi^{n-i} \;\omega^{i-3} \quad D''^{i}f = 0,$$

$$\sum_{i=1}^{i=n-1} \frac{1}{1.2.3\ldots i}.\,\psi^{n-i-1}\omega^{i-1} D' D''^{i}f = 0,$$

to which is to be added $f=0$. From these four equations the four unknown quantities $\psi : \omega$, $w : x : y : z$ are to be determined and the extraneous solutions rejected. It is of course intended that p, q, r, s, which denote the first derived functions of f, should be replaced by their values. In order to give an idea how numerous the extraneous solutions may be, I may mention that for $n = 3$, the system reduces itself to $f=0$, $D''^2f=0$, $D''^3f=0$, and that all the 90 solutions are extraneous, inasmuch as 18 solutions belong to the system (to be taken six times over) $w=0$, $x=0$, $f=0$, and 72 to the system (to be taken six times over) $r=0$, $s=0$, $f=0$.

In order to exhaust the singular tangent planes of a general (as regard order) surface, we must imagine the planes which touch the basis along the curve $f=0$, $\nabla f=0$, consequently in curves having a cusp at the point of contact, such planes, considered in respect to class, have two coincident points of contact, and are therefore singular tangent planes. The system of the planes in question generate what I call the osculating circumscribed developable, the curve in question may be called the osculation curve; it separates the region of the basis where the measure of curvature is negative (consequently where ∇f is positive and the two tangents at the double point of the contact section are real) from the region where the measure of curvature is positive. There are certain determinate points of the basis where the osculation curve and the curve of contact of the doubly circumscribed developable, 1° simply intersect, 2° touch. A plane which touches the basis at a point of the former kind intersects the basis in a curve having a double point and also a cusp; a plane touching the basis at a point of the latter kind cuts the basis in a curve having at the point of contact a self-touching double point, that is, a double point where the two branches touch; the tangent at such double point coincides with that of the osculation curve; and if in the neighbourhood of such a point we follow the motion of the double tangent plane, we find that upon one side of the curve of osculation the two points of contact of the plane are real points indefinitely near to each other, and on the other side the plane is still real but the two points of contact are imaginary and conjugate to each other.

With respect to these singular developables and planes I assume the numerical relations following:

1°. $a = \frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)$ the class of the doubly circumscribed developable, A the (still unknown) order.

2°. $b = 4n(n-1)(n-2)$ the class of the osculating circumscribed developable $B = 2n(n-2)(3n-4)$ its order.

3°. κ the (still unknown) number of the triple tangent planes.

4°. $\lambda = 4n(n-2)(n-3)(n^3+3n-16)$ the number of planes touching the surface in a curve having a double point and also a cusp.

5°. $\mu = 2n(n-2)(11n-24)$ the number of planes touching the surface in a curve having a self-touching double point.

The class of the surface is $k = n(n-1)^2$. If the surface were general (as regards class) the order would be $k(k-1)^2$. The difference $k(k-1)^2 - n$ is to be accounted for by means of the singular developables and tangent planes. The doubly circumscribed developable in itself (abstracting the tangent planes of a higher singularity included in it) diminishes the class of the surface by $ak + 2A$, the osculating circumscribed developable (with the like abstraction) diminishes the class by $2bk + 3B$, each triple tangent plane (abstracting the three sheets of the developable to which it is common) diminishes the class by 3, each tangent plane cutting the surface in a curve having a double point and cusp by 4, and lastly each tangent plane cutting the surface in a curve having a self-touching double point by 6. We have thus

$$(a+2b)\,k + 2A + 3B + 3\kappa + 4\lambda + 6\mu = k\,(k-1)^2 - n,$$

which gives between the still unknown numbers A and κ the following relation:

$$2A+3\kappa=\tfrac{1}{2}n(n-2)(n^7-4n^6+7n^5-45n^4+118n^3-115n^2+508n-912).$$

For $n = 3$ we have

$$a = 27, \quad b = 30, \quad B = 24, \quad \lambda = 0, \quad \mu = 54, \quad k = 12.$$

But as a curve of the third order cannot have two double points without breaking up into a conic and a right line, it is clear that the doubly circumscribed developable of a surface of the third degree can consist only of planes passing through fixed lines upon the basis, and that since the class is $a = 27$, there are upon the basis 27 such lines which play the part of the developable in question. But as these lines are not in general intersected by an arbitrary line, we must have $A = 0$ for the degree of this degenerate developable and the formula gives $\kappa = 45$ as the number of the triple tangent planes, which it is clear meet the basis in three right lines, a number which may be obtained by other considerations.

Remark by the Translator. The investigations contained in the present portion of Prof. Schläfli's Memoir, with respect to the general theory of algebraical surfaces, are similar in character to those of Mr. Salmon, and several of the author's results have been already given in Mr. Salmon's Memoirs in the *Journal*, but the theory is here carried a few steps further than in the memoirs just referred to; and the knowledge which I have of Mr. Salmon's still unpublished Memoir on Reciprocal Surfaces, in which the whole subject is considered in a more complete manner (and in particular formulæ are given leading to the determination of the two numbers A and κ) was clearly not a reason for delaying the publication of Prof. Schläfli's interesting Memoir, which was kindly sent by him for insertion in the *Journal*.

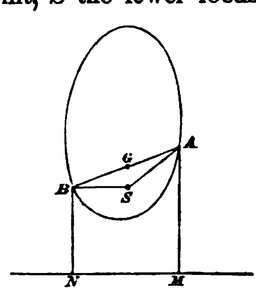
(*To be Continued*)

SOLUTION OF A MECHANICAL PROBLEM.

A UNIFORM rod is constrained to slide with its extremities on a conic section, whose axis major is vertical, and whose latus-rectum is less than the length of the rod: find the position of stable equilibrium.

This problem admits of a very simple solution, depending on the principle that for a position of stable equilibrium, the height of the centre of gravity must be a minimum.

Let AB be the rod, G its middle point, S the lower focus of the conic section, draw BN, AM vertical, meeting the lower direction in M, N. Let e be the eccentricity of the conic section.



Then the height of G above the direction

$$= \tfrac{1}{2}(AM + BN)$$

$$= \frac{1}{2e}(AS + BS),$$

and the height of G will therefore be least when $AS + BS$ is least, that is, when A and B both lie in a straight line passing through S, or when the rod passes through the focus of the conic section.

H. G.

ON SIR W. R. HAMILTON'S METHOD FOR THE PROBLEM OF THREE OR MORE BODIES.

By A. CAYLEY.

THE problem of three or more bodies is considered by Sir W. R. Hamilton in his two well known memoirs on a general method in Dynamics, *Phil. Trans.* 1834 and 1835, and the differential equations for the relative motion with respect to the central body of all the other bodies are obtained in a form containing a single disturbing function only. Several methods of integration are given or indicated, among others, one which is in fact the method of the variation of the elements as applied to the particular form of the equations of motion. But the investigation shews (and Sir W. R. Hamilton notices this as a defect in his theory, as compared with the ordinary theory of the variation of the elements), that in the method in question, the elements are not osculating elements, i.e. that the positions only, and not the velocities of the bodies can be calculated, as if the elements remained constant during an element of time. The peculiar advantage of the method is of course the having a single disturbing function only, and this seems so important, that if I may venture to express an opinion, I cannot but think that the method will ultimately be employed for the purposes of Physical astronomy. But, however this may be, it has appeared to me that it may be useful to present the method in a separate and distinct form, disengaged from the general theory as an illustration of which it was given by the author; and this is what I propose now to do.

Consider a central body M, and two other bodies M_1, M_2, and let the coordinates of M referred to a fixed origin be x, y, z, and the coordinates of M_1, M_2 referred to the body M as origin be x_1, y_1, z_1 and x_2, y_2, z_2 respectively. Then the coordinates of M_1, M_2 referred to the fixed origin, are $x+x_1$, $y+y_1$, $z+z_1$ and $x+x_2$, $y+y_2$, $z+z_2$ respectively, and if as usual T denotes the Vis-viva or half sum of each mass into the square of its velocity, and U denote the force function, then we have

$$\begin{aligned} T = {} & \tfrac{1}{2}M\,(x'^2+y'^2+z'^2), \\ & + \tfrac{1}{2}M_1\,\{(x'+x_1')^2+(y'+y_1')^2+(z'+z_1')^2\} \\ & + \tfrac{1}{2}M_2\,\{(x'+x_2')^2+(y'+y_2')^2+(z'+z_2')^2\}, \end{aligned}$$

$$\begin{aligned} U = {} & \frac{MM_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}} \\ & + \frac{MM_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}} \\ & + \frac{M_1M_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}, \end{aligned}$$

and the equations of motion are as usual

$$\frac{d}{dt}\frac{dT}{dx'}-\frac{dT}{dx}=\frac{dU}{dx},$$

&c.

If we assume that the centre of gravity of the bodies is at rest, then we have

$$Mx'+M_1\,(x'+x_1')+M_2\,(x'+x_2')=0,\ \&c.,$$

and consequently

$$x' = -\frac{M_1x_1' + M_2x_2'}{M + M_1 + M_2},\ y' = -\frac{M_1y_1' + M_2y_2'}{M + M_1 + M_2},\ z' = -\frac{M_1z_1' + M_2z_2'}{M + M_1 + M_2}.$$

Now the value of T is

$$\begin{aligned} T = {} & \tfrac{1}{2}(M + M_1 + M_2)(x'^2 + y'^2 + z'^2) \\ & + x'(M_1x_1' + M_2x_2') + y'(M_1y_1' + M_2y_2') + z'(M_1z_1' + M_2z_2') \\ & + \tfrac{1}{2}M_1(x_1'^2 + y_1'^2 + z_1'^2) \\ & + \tfrac{1}{2}M_2(x_2'^2 + y_2'^2 + z_2'^2), \end{aligned}$$

or putting for x', y', z' their values

$$\begin{aligned} T = {} & \tfrac{1}{2}M_1(x_1'^2 + y_1'^2 + z_1'^2) \\ & + \tfrac{1}{2}M_2(x_2'^2 + y_2'^2 + z_2'^2) \\ & - \tfrac{1}{2}\frac{1}{M + M_1 + M_2} \times \\ & \qquad \{(M_1x_1' + M_2x_2')^2 + (M_1y_1' + M_2y_2')^2 + (M_1z_1' + M_2z_2')^2\}, \end{aligned}$$

and with this new value of T the equations of motion still are

$$\frac{dT}{dx_1'} - \frac{dT}{dx_1} = \frac{dU}{dx_1},\ \&c.$$

Suppose now that the differential coefficients of T, with respect to x_1', y_1', z_1'; x_2', y_2', z_2', are respectively P_1, Q_1, R_1; P_2, Q_2, R_2, i.e. write

$$\frac{dT}{dx_1'} = P_1,\ \&c.,$$

and imagine T expressed as a function of P_1, Q_1, R_1; P_2, Q_2, R_2, and when this is done put $H = T - U$ (so that H stands for a function of P_1, Q_1, R_1; P_2, Q_2, R_2; x_1, y_1, z_1; x_2, y_2, z_2), then the equations of motion in Sir W. R. Hamilton's form are

$$\frac{dx_1}{dt} = \frac{dH}{dP_1},\quad \frac{dP_1}{dt} = -\frac{dH}{dx_1},\ \&c.$$

Now from the last given value of T

$$P_1 = M_1x_1' - \frac{M_1}{M + M_1 + M_2}(M_1x_1' + M_2x_2'),$$

$$P_2 = M_2x_2' - \frac{M_2}{M + M_1 + M_2}(M_1x_1' + M_2x_2'),$$

and thence

$$P_1 + P_2 = \frac{M}{M + M_1 + M_2}(M_1x_1' + M_2x_2'),$$

and consequently

$$M_1 x_1' = P_1 + \frac{M_1}{M}(P_1 + P_2),$$

$$M_2 x_2' = P_2 + \frac{M_2}{M}(P_1 + P_2),$$

and we have

$$T = \frac{1}{2M_1}\left[\left\{P_1 + \frac{M_1}{M}(P_1 + P_2)\right\}^2 + \left\{Q_1 + \frac{M_1}{M}(Q_1 + Q_2)\right\}^2 + \left\{R_1 + \frac{M_1}{M}(R_1 + R_2)\right\}^2\right]$$
$$+ \frac{1}{2M_2}\left[\left\{P_2 + \frac{M_2}{M}(P_1 + P_2)\right\}^2 + \left\{Q_2 + \frac{M_2}{M}(Q_1 + Q_2)\right\}^2 + \left\{R_2 + \frac{M_2}{M}(R_1 + R_2)\right\}^2\right]$$
$$- \frac{1}{2M^2}(M + M_1 + M_2)\left[(P_1 + P_2)^2 + (Q_1 + Q_2)^2 + (R_1 + R_2)^2\right],$$

or reducing

$$T = \left(\frac{1}{2M_1} + \frac{1}{2M}\right)(P_1^2 + Q_1^2 + R_1^2)$$
$$+ \left(\frac{1}{2M_2} + \frac{1}{2M}\right)(P_2^2 + Q_2^2 + R_2^2)$$
$$+ \frac{1}{M}(P_1 P_2 + Q_1 Q_2 + R_1 R_2),$$

and consequently

$$H = \left(\frac{1}{2M_1} + \frac{1}{2M}\right)(P_1^2 + Q_1^2 + R_1^2)$$
$$+ \left(\frac{1}{2M_2} + \frac{1}{2M}\right)(P_2^2 + Q_2^2 + R_2^2)$$
$$+ \frac{1}{M}(P_1 P_2 + Q_1 Q_2 + R_1 R_2)$$
$$- \frac{MM_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}$$
$$- \frac{MM_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}}$$
$$- \frac{M_1 M_2}{\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}}},$$

and H having this value, the equations of motion are as before mentioned

$$\frac{dx_1}{dt}=\frac{dH}{dP_1},\quad \frac{dP_1}{dt}=-\frac{dH}{dx_1},\ \&c.$$

Instead of H write $H+\Upsilon$ where

$$\begin{aligned}H=&\left(\frac{1}{2M_1}+\frac{1}{2M}\right)(P_1^2+Q_1^2+R_1^2)\\&+\left(\frac{1}{2M_2}+\frac{1}{2M}\right)(P_2^2+Q_2^2+R_2^2)\\&-\frac{MM_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}\\&-\frac{MM_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}},\end{aligned}$$

and

$$\begin{aligned}\Upsilon=&\frac{1}{M}(P_1P_2+Q_1Q_2+R_1R_2)\\&-\frac{M_1M_2}{\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}}},\end{aligned}$$

and the function Υ is to be treated as a disturbing function. The equations of motion for the body M_1 become

$$\frac{dx_1}{dt}=\frac{M+M_1}{MM_1}P_1+\frac{d\Upsilon}{dP_1},\quad \frac{dP_1}{dt}=-\frac{MM_1x_1}{(x_1^2+y_1^2+z_1^2)^{\frac{3}{2}}}-\frac{d\Upsilon}{dx_1},$$

$$\frac{dy_1}{dt}=\frac{M+M_1}{MM_1}Q_1+\frac{d\Upsilon}{dQ_1},\quad \frac{dQ_1}{dt}=-\frac{MM_1y_1}{(x_1^2+y_1^2+z_1^2)^{\frac{3}{2}}}-\frac{d\Upsilon}{dy_1},$$

$$\frac{dz_1}{dt}=\frac{M+M_1}{MM_1}R_1+\frac{d\Upsilon}{dR_1},\quad \frac{dR_1}{dt}=-\frac{MM_1z_1}{(x_1^2+y_1^2+z_1^2)^{\frac{3}{2}}}-\frac{d\Upsilon}{dz_1},$$

and there is of course a precisely similar system of equations of motion for the body M_2.

If we neglect Υ the left hand equations shew that P_1, Q_1, R_1 denote the velocities or differential coefficients $\frac{dx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{dz_1}{dt}$ multiplied by the constant factor $\frac{MM_1}{M+M_1}$, and substituting these values in the right hand equations, we obtain the ordinary equations for the elliptic motion of the body M_1; and similarly for the body M_2. We may, if we please, complete the solution by the method of the variation of the arbitrary constants. Suppose for this purpose that $a_1, b_1, c_1, e_1, f_1, g_1$ are the elements for the elliptic motion of the body M_1, then treating these elements as variable we must have

$$\frac{dx_1}{da_1}\frac{da_1}{dt} + \frac{dx_1}{db_1}\frac{db_1}{dt} \dots + \frac{dx_1}{dg_1}\frac{dg_1}{dt} = \frac{d\Upsilon}{dP_1}, \text{ \&c.,}$$

$$\frac{dP_1}{da_1}\frac{da_1}{dt} + \frac{dP_1}{db_1}\frac{db_1}{dt} \dots + \frac{dP_1}{dg_1}\frac{dg_1}{dt} = -\frac{d\Upsilon}{dx_1},$$

and it appears from these equations that as already noticed the disturbed values of the velocities are not (as they are in the ordinary theory) identical with the undisturbed values.

The disturbing function Υ may be considered as a function of the elements of the two orbits and of the time, and it is easy to obtain, as in the ordinary theory, the values of the differential coefficients $\frac{da_1}{dt}$, &c. in the form

$$\frac{da_1}{dt} = (a_1, b_1)\frac{d\Upsilon}{db_1} + (a_1, c_1)\frac{d\Upsilon}{dc_1} \dots + (a_1, g_1)\frac{d\Upsilon}{dg_1},$$

where $$(a_1, b_1) = \frac{\delta(a_1, b_1)}{\delta(x_1, P_1)} + \frac{\delta(a_1, b_1)}{\delta(y_1, Q_1)} + \frac{\delta(a_1, b_1)}{\delta(z_1, R_1)},$$

if for shortness

$$\frac{\delta(a_1, b_1)}{\delta(x_1, P_1)} = \frac{da_1}{dx_1}\frac{db_1}{dP_1} - \frac{da_1}{dP_1}\frac{db_1}{dx_1}.$$

It will be remembered that in the ordinary theory, if Ω denote Lagrange's disturbing function ($\Omega = -R$ if R is the disturbing function of the Mecanique Celeste) the corresponding formulæ are

$$\frac{da}{dt} = (a, b)\frac{d\Omega}{db} + (a, c)\frac{d\Omega}{dc} \dots + (a, g)\frac{d\Omega}{dg},$$

where $$(a, b) = \frac{\delta(a, b)}{\delta(x', x)} + \frac{\delta(a, b)}{\delta(y', y)} + \frac{\delta(a, b)}{\delta(z', z)},$$

if for shortness

$$\frac{\delta(a, b)}{\delta(x', x)} = \frac{da}{dx'}\frac{db}{dx} - \frac{da}{dx}\frac{db}{dx'},$$

or, what is the same thing, where

$$(a, b) = -\frac{\delta(a, b)}{\delta(x, x')} - \frac{\delta(a, b)}{\delta(y, y')} - \frac{\delta(a, c)}{\delta(z, z')},$$

and $$\frac{\delta(a, b)}{\delta(x, x')} = \frac{da}{dx}\frac{db}{dx'} - \frac{da}{dx'}\frac{db}{dx}.$$

Now the values of the coefficients (a_1, b_1), &c. depend merely on the form of the expressions for a_1, b_1, &c. in terms of P_1, Q_1, R_1, x_1, y_1, z_1 and t; hence comparing the two systems of formulæ and observing P_1, Q_1, R_1 (which in the formulæ for the present theory correspond with x_1', y_1', z_1' in the other system of formulæ) are respectively equal to x_1', y_1', z_1', each of them multiplied by the constant factor $\dfrac{MM_1}{M+M_1}$, it is easy to see that the formulæ for the variations of any given system of elements in the present theory are at once deduced from the formulæ for the variations of the same system of elements in the ordinary theory by writing $-\Upsilon$ in the place of Ω and multiplying the values of the variations by the constant factor $\dfrac{MM_1}{M+M_1}$.

Take then as elements Jacobi's canonical system,* viz. if we put

a_1 the semiaxis major,

e_1 the excentricity,

ϖ_1 the longitude in orbit of pericentre,

ϵ_1 the mean longitude in orbit at epoch,

θ_1 the longitude of node,

ϕ_1 the inclination,

and n_1 the mean motion $\left\{= \sqrt{\left(\dfrac{M+M_1}{a_1^3}\right)}\right\}$,

then the canonical elements are

$$\begin{aligned}
\mathfrak{A}_1 &= -\tfrac{1}{2}n_1^2a_1^2,\\
\mathfrak{B}_1 &= n_1a_1\sqrt{(1-e_1^2)},\\
\mathfrak{C}_1 &= n_1a_1\sqrt{(1-e_1^2)}\cos\phi_1,\\
\mathfrak{F}_1 &= \frac{1}{n_1}(\epsilon_1-\varpi_1),\\
\mathfrak{G}_1 &= \varpi_1-\theta_1,\\
\mathfrak{H}_1 &= \theta_1,
\end{aligned}$$

* I have for uniformity adopted Jacobi's canonical system, see his paper "Neues Theorem der analytischen Mechanik," *Crelle*, t. xxx. pp. 117–120 (1846); but it is proper to remark that Sir W. R. Hamilton, in his Memoirs above referred to, employs a slightly different but equally elegant system of canonical elements, and that the discovery of such a system belongs to Sir W. R. Hamilton and is part of his general theory.

(the sign of the two elements $\mathfrak{A}_1$, $\mathfrak{F}_1$ has been changed, but this makes no difference in the formulæ) then the equations for the variations of the elements are

$$\frac{d\mathfrak{A}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{F}_1},$$

$$\frac{d\mathfrak{B}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{G}_1},$$

$$\frac{d\mathfrak{C}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{H}_1},$$

$$\frac{d\mathfrak{F}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{A}_1},$$

$$\frac{d\mathfrak{G}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{B}_1},$$

$$\frac{d\mathfrak{H}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{C}_1},$$

and it is easy thence to deduce the formulæ for the variations of any system of elements which it may be thought proper to make use of, for instance the system a_1, e_1, ϖ_1, ϵ_1, θ_1, ϕ_1.

It will be recollected that in the preceding system of formulæ the value of the disturbing function Υ is

$$\Upsilon = \frac{1}{M}(P_1P_2 + Q_1Q_2 + R_1R_2) - \frac{M_1M_2}{\sqrt{\{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\}}},$$

and that as a first approximation P_1, Q_1, R_1 are respectively equal to the velocities x_1', y_1', z_1', each multiplied by the constant factor $\frac{MM_1}{M+M_1}$, and P_2, Q_2, R_2 are respectively equal to the velocities x_2', y_2', z_2', each multiplied by $\frac{MM_2}{M+M_2}$.

2, Stone Buildings,
18*th Oct.*, 1856.

ON THE GEOMETRICAL INTERPRETATION OF THE EXPRESSION $rt - s^2$.

By H. W. Elphinstone.

In Gregory's Solid Geometry the following remark is stated to be due to Mr. Cayley: viz. that the curve of intersection of a surface with its tangent plane has generally a double point at the point of contact. This remark is more fertile of consequences than may at first sight be evident.

Let $$z = f(x, y) \text{ ....(1)}$$

be the equation to a surface, and let

$$\zeta - z = p_1(\xi - x) + q_1(\eta - y) \text{ .... (2)}$$

be the equation to the tangent plane at the point xyz on the surface in which p_1, q_1 are the particular values of $\frac{dz}{dx}$, $\frac{dz}{dy}$ at the point in question.

The curve of intersection of the tangent plane and its surface is obtained by combining equation (2) with the equation

$$\zeta = f(\xi, \eta) \text{ ....(3).}$$

Let us find the values of $\frac{d\xi}{d\eta}$ belonging to the curve of intersection at the point of contact. Differentiating (2) and (3), we have

from (2) $$d\zeta = p_1 d\xi + q_1 d\eta \text{ ....(4),}$$

from (3) $$d\zeta = p\, d\xi + q\, d\eta \text{ ....(5),}$$

where in (5) we must substitute p_1, q_1 for p and q respectively after differentiation. On attempting to find the value of $\frac{d\xi}{d\eta}$ from equations (4) and (5) it appears under the form $\frac{0}{0}$.

Differentiating again, we have

from (4) $$d^2\zeta = p_1 d^2\xi + q_1 d^2\eta \text{ ....(6),}$$

from (5) $$d^2\zeta = p\, d^2\xi + q\, d^2\eta + r d\xi^2 + 2s d\xi d\eta + t d\eta^2 \text{ .... (7),}$$

where for p, q, r, s, t we must substitute their values at the point in question. Subtracting equation (6) from (7) we arrive at the equation

$$r\left(\frac{d\xi}{d\eta}\right)^2 + 2s\frac{d\xi}{d\eta} + t = 0 \text{ .... (8),}$$

for determining the values of $\frac{d\xi}{u\eta}$ at the double point in question.

The roots of (8) are real and unequal, real and equal, or imaginary, as

$$s^2 - rt > = < 0 \text{ .... } (9),$$

a well known condition for determining the form of the surface at the point in question.

1st. Let $\qquad s^2 - rt > 0.$

The roots of equation (8) are real and unequal; consequently the tangent plane cuts the surface along two lines in the neighbourhood of the point of contact. The 4 parts into which the tangent plane thus divides the surface will be alternately above and below the tangent plane

above
below × below. A saddle will afford a familiar example.
above

2nd. Let $\qquad s^2 - rt < 0.$

Here the roots of (8) are imaginary; consequently the surface is not cut by its tangent plane in the neighbourhood of the point, it therefore lies wholly on one side of it, or is convex.

3rd. Let $\qquad s^2 - rt = 0.$

This may occur for three different reasons.

1st. r, s, t may all vanish at the point in question. In this case we have to differentiate again in order to find the values of $\frac{d\xi}{d\eta}$, the point on the curve of intersection becomes a triple point at least, and the point on the surface becomes a singular point.

2nd. $s^2 - rt$ may be rendered $= 0$ by the existence of some factor in both p and q which vanishes at the point on contact. This indicates a ridge on the surface, and is treated of in all the elementary works on the Differential Calculus.

3rd. $s^2 - rt$ may equal zero without satisfying either of the above mentioned conditions. The interpretation (which is generally slurred over in works on Geometry) may be obtained without much trouble from the expression for the radius of curvature of a normal section to a surface

$$R = \frac{k}{rm^2 + 2sm + t}.$$

In this case, since the two values of $\frac{d\xi}{d\eta}$ are equal, the two branches of the curve of intersection touch at the point. And since the denominator of R becomes $=0$, when the normal section is taken along the common tangent, it follows that that section has a point of inflection at that point. Consequently the form of the surface in the neighbourhood of the point may be represented as follows: take any plane curve with a point of inflection in it, and let two plane curves, having a common tangent, move with their point of contact on the first mentioned curve, so that their common tangent may coincide with the tangent to the other curve at the point of inflection. It may however happen that the normal section, instead of having a point of inflection, becomes a straight line. This is a line along which the surface may be bent; and if such a line occurs at every point of the surface, the surface may be bent along such lines in succession, till every element is in the same plane as the succeeding one; or, in other words, till the surface is plane. Such surfaces are known under the name "developable," and at every point satisfy the condition $s^2 - rt = 0$.

ON LAGRANGE'S SOLUTION OF THE PROBLEM OF TWO FIXED CENTRES.

By A. Cayley.

The following variation of Lagrange's Solution of the Problem of Two Fixed Centres,* is, I think, interesting, as showing more distinctly the connection between the differential equations and the integrals. The problem referred to is as follows: viz. to determine the motion of a particle acted upon by forces tending to two fixed centres, such that r, q being the distances of the particle from the two centres respectively, and α, β, γ being constants, the forces are $\frac{\alpha}{r^2} + 2\gamma r$ and $\frac{\beta}{q^2} + 2\gamma q$.

Take the first centre as origin and the line joining the two centres as axis of x; and let h be the distance between the two centres, then writing for symmetry

$$x = x_1 = x_2 + h,$$

* Lagrange's Solution was first published in the *Anciens Mem. de Turin*, t. IV. and is reproduced in the *Mecanique Analytique*.

(so that x_1 is the coordinate corresponding to the first centre as origin, and x_2 the coordinate corresponding to the second centre as origin) the distances are given by the equations

$$r^2 = x_1^2 + y^2 + z^2, \quad q^2 = x_2^2 + y^2 + z^2,$$

and the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{\alpha x_1}{r^3} - \frac{\beta x_2}{q^3} - 2\gamma(x_1 + x_2),$$

$$\frac{d^2y}{dt^2} = -\frac{\alpha y}{r^3} - \frac{\beta y}{q^3} - 4\gamma y,$$

$$\frac{d^2z}{dt^2} = -\frac{\alpha z}{r^3} - \frac{\gamma z}{q^3} - 4\gamma z,$$

and we obtain at once the integral of Vis-viva, viz. multiplying the three equations by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, adding and integrating $\left(\text{observing that } \frac{dx}{dt} = \frac{dx_1}{dt} = \frac{dx_2}{dt}\right)$ we have

$$\tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \frac{\alpha}{r} + \frac{\beta}{q} - \gamma(r^2 + q^2) + 2H \ldots (1, a),$$

and with equal facility, the equation of areas round the line joining the two centres, viz. multiplying the second and third equations by $-z$, y, adding and integrating, we have

$$y\frac{dz}{dt} - z\frac{dy}{dt} = B \ldots\ldots\ldots\ldots\ldots\ldots (2, a).$$

So far Lagrange: to obtain a third integral I form the equation

$$\begin{aligned}
&\left[-2(y^2+z^2)\frac{dx}{dt} + (x_1+x_2)\left(y\frac{dy}{dt} + z\frac{dz}{dt}\right)\right] \times \\
&\qquad\qquad \left\{\frac{d^2x}{dt^2} + \frac{\alpha x_1}{r^3} + \frac{\beta x_2}{q^3} + 2\gamma(x_1+x_2)\right\} \\
+&\left[(x_1+x_2)\,y\frac{dx}{dt} - 2x_1x_2\frac{dy}{dt}\right] \times \\
&\qquad\qquad \left\{\frac{d^2y}{dt^2} + \frac{\alpha y}{r^3} + \frac{\beta y}{q^3} + 4\gamma y\right\} \\
+&\left[(x_1+x_2)\,z\frac{dx}{dt} - 2x_1x_2\frac{dz}{dt}\right] \times \\
&\qquad\qquad \left\{\frac{d^2z}{dt^2} + \frac{\alpha z}{r^3} + \frac{\beta z}{q^3} + 4\gamma z\right\} = 0.
\end{aligned}$$

The terms, independent of the forces, are

$$-2(y^2+z^2)\frac{dx}{dt}\frac{d^2x}{dt^2}+(x_1+x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\frac{d^2x}{dt^2}$$

$$+(x_1+x_2)\frac{dx}{dt}\left(y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}\right)-2x_1x_2\left(\frac{dy}{dt}\frac{d^2y}{dt^2}+\frac{dz}{dt}\frac{d^2z}{dt^2}\right),$$

which are equal to

$$\frac{d}{dt}\left[-(y^2+z^2)\left(\frac{dx}{dt}\right)^2+(x_1+x_2)\frac{dx}{dt}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)-x_1x_2\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}\right],$$

and the terms depending on the forces are readily reduced to the form

$$\frac{d}{dt}\left\{-\frac{h\alpha x_1}{r}+\frac{h\beta x_2}{q}+h^2\gamma(y^2+z^2)\right\},$$

in fact, considering first the terms multiplied by α, these are

$$\frac{x_1}{r^3}\left\{-2(y^2+z^2)\frac{dx}{dt}+(x_1+x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$+\frac{1}{r^3}\left\{(x_1+x_2)(y^2+z^2)\frac{dx}{dt}-2x_1x_2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\},$$

which is equal to

$$\frac{1}{r^3}\left\{(x_2-x_1)(y^2+z^2)\frac{dx}{dt}+x_1(x_1-x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{1}{r^3}\left\{-h(y^2+z^2)\frac{dx}{dt}+hx_1\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{h}{r^3}\left\{-(x_1^2+y^2+z^2)\frac{dx_1}{dt}+x_1\left(x\frac{dx_1}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{h}{r^3}\left\{-r^2\frac{dx_1}{dt}+rx_1\frac{dr}{dt}\right\}$$

$$=-h\frac{d}{dt}\frac{x_1}{r},$$

and similarly the term multiplied by β is

$$h\frac{d}{dt}\frac{x_2}{q},$$

lastly, the term multiplied by γ is

$$-4(x_1+x_2)(y^2+z^2)\frac{dx}{dt}+2(x_1+x_2)^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$+4(x_1+x_2)(y^2+z^2)\frac{dx}{dt}- \quad 8x_1x_2 \quad \left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=2(x_1-x_2)^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=2h^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\frac{d}{dt}h^2(y^2+z^2).$$

The preceding combination of the differential equations gives therefore an equation integrable per se, and effecting the integration we have

$$-(y^2+z^2)\left(\frac{dx}{dt}\right)^2+(x_1+x_2)\frac{dx}{dt}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)-x_1x_2\left\{\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$

$$-\frac{h\alpha x_1}{r}+\frac{h\beta x_2}{q}+h^2\gamma(y^2+z^2)=\mathrm{K}\ldots\ldots\ldots(3,\ a),$$

which is the third integral equation. It may be convenient to mention here (what appears by the comparison of the formulæ obtained in the sequel with the corresponding formulæ of Lagrange) that the value of Lagrange's constant of integration C is

$$C=\mathrm{K}-2Hh^2-B^2+\tfrac{1}{4}\gamma h^4.$$

Making use of the ordinary transformation

$$(y^2+z^2)\left\{\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2+\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)^2,$$

the integral equations may be written under the forms

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2+\frac{1}{2(y^2+z^2)}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2$$

$$=\frac{\alpha}{r}+\frac{\beta}{q}-\gamma(r^2+q^2)+2H-\frac{B^2}{2(y^2+z^2)}\ldots\ldots(1,\ b),$$

$$y\frac{dz}{dt}-z\frac{dy}{dt}=B\ldots\ldots\ldots\ldots\ldots\ldots(2,\ b),$$

$$-(y^2+z^2)\left(\frac{dx}{dt}\right)^2+(x_1+x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\frac{dx}{dt}-\frac{x_1x_2}{y^2+z^2}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2$$

$$-\frac{h\alpha x_1}{r}+\frac{h\beta x_2}{q}+h^2\gamma(y^2+z^2)=\mathrm{K}+\frac{B^2x_1x_2}{y^2+z^2}\;\ldots\ldots\;(3,\,b),$$

and observing that y^2+z^2, x_1, x_2 are in fact functions of r, q, it is clear that the determination of r, q in terms of t depends upon the first and third equations alone. Moreover the form of the equations shews that we can at once eliminate dt and thus obtain a differential equation between r, q alone. It would be difficult to discover *à priori* before actually obtaining the differential equation in question, that it would be possible to effect the separation of the variables, but we know that this can be done by taking instead of r, q the new variables $u=r+q$, $s=r-q$. In order to complete the solution the first step is to introduce the variables r, q into the first and third equations: for this purpose we have

$$x_1=\frac{h^2+r^2-q^2}{2h},\quad -x_2=\frac{h^2-r^2+q^2}{2h},\quad x_1+x_2=\frac{r^2-q^2}{h},$$

$$y^2+z^2=\frac{\nabla}{4h^2},$$

if for shortness

$$\nabla=2r^2q^2+2h^2r^2+2h^2q^2-h^4-r^4-q^4,$$

and consequently

$$\frac{dx}{dt}=\frac{1}{h}\left(r\frac{dr}{dt}-q\frac{dq}{dt}\right),$$

$$y\frac{dy}{dt}+z\frac{dz}{dt}=\frac{1}{2h^2}\left\{(h^2-r^2+q^2)\,r\frac{dr}{dt}+(h^2+r^2-q^2)\,q\frac{dq}{dt}\right\}.$$

Substituting these values in the two equations, we find

$$\nabla(r\,dr-q\,dq)^2+\{(h^2-r^2+q^2)\,r\,dr+(h^2+r^2-q^2)\,q\,dq\}^2$$

$$=2h^2\left\{\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma(r^2+q^2)\nabla+2H\nabla-2h^2B^2\right\}dt^2\;\ldots\;(1,\,c),$$

$$-\nabla^2(r\,dr-q\,dq)^2$$

$$+2\nabla(r\,dr-q\,dq)(r^2-q^2)\{(h^2-r^2+q^2)\,r\,dr+(h^2+r^2-q^2)\,q\,dq\}$$

$$+\{h^4-(r^2-q^2)^2\}\{(h^2-r^2+q^2)\,r\,dr+(h^2+r^2-q^2)\,q\,dq\}^2$$

$$=h^4\left[\frac{2\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{2\beta\nabla}{q}(h^2-r^2+q^2)-\gamma\nabla^2\right.$$

$$\left.+4\mathrm{K}\nabla-4B^2\{h^4-(r^2-q^2)^2\}\right]dt^2\;\ldots\ldots\;(2,\,c).$$

The first equation is easily reduced to

$$4h^2\{r^2q^2(dr^2+dq^2)+(h^2-r^2-q^2)\,rq\,dr\,dq\}$$
$$=2h^2\left\{\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma(r^2+q^2)\nabla+2H\nabla-2h^2B^2\right\}dt^2,$$

the second equation gives

$$h^4\{(h^2-r^2+q^2)\,r\,dr+(h^2+r^2-q^2)\,q\,dq\}^2$$
$$-h^4\{(h^2-r^2-3q^2)\,r\,dr-(h^2-3r^2+q^2)\,q\,dq\}^2$$
$$=h^4\left[\frac{2\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{2\beta\nabla}{q}(h^2-r^2+q^2)-\gamma\nabla^2\right.$$
$$\left.+4\mathrm{K}\nabla-4B^2\{h^4-(r^2-q^2)^2\}\right]dt^2,$$

and the function on the left-hand side is

$$8h^4(h^2-r^2-q^2)q^2r^2(dr^2+dq^2)+4h^4\{(h^2-r^2-q^2)^2+4q^2r^2\}\,rq\,dr\,dq.$$

Hence putting for a moment

$$M=\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma\nabla(r^2+q^2)+2H\nabla-2h^2B^2,$$
$$N=\frac{\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{\beta\nabla}{q}(h^2-r^2+q^2)-\tfrac{1}{2}\gamma\nabla^2$$
$$+2\mathrm{K}\nabla-2B^2\{h^4-(r^2-q^2)^2\},$$

we have

$$2r^2q^2(dr^2+dq^2)+\qquad(h^2-r^2-q^2)\,2rq\,dr\,dq=Mdt^2,$$
$$2(h^2-r^2-q^2)\,2r^2q^2(dr^2+dq^2)+\{(h^2-r^2-q^2)^2+4q^2r^2\}\,2rq\,dr\,dq=Ndt^2,$$

and thence recollecting that

$$-(h^2-r^2-q^2)^2+4q^2r^2=\nabla,$$

we find

$$\nabla 2rq\,dr\,dq=\{(h^2-r^2-q^2)\,2M-N\}\,dt^2,$$
$$\nabla 2r^2q^2(dr^2+dq^2)=[-\{(h^2-r^2-q^2)^2+4q^2r^2\}M+(h^2-r^2-q^2)N]dt^2,$$

and substituting for M, N their values, the functions on the right-hand side contain ∇ as a factor, and dividing by ∇, we obtain

$$2rq\,dr\,dq=\left[\frac{\alpha}{r}(3r^2+q^2-h^2)+\frac{\beta}{q}(r^2+3q^2-h^2)\right.$$
$$-\tfrac{1}{2}\gamma(3r^4+3q^4+10q^2r^2-2h^2r^2-2h^2q^2-h^4)$$
$$\left.+4H(q^2+r^2-h^2)+2\mathrm{K}-2B^2\right]dt^2\ \ldots\ldots\ldots\ldots\ (1,d),$$

$$2r^2q^2(dr^2+dq^2)=[2\alpha r(r^2+3q^2-h^2)+2\beta q(3r^2+q^2-h^2)$$
$$-\tfrac{1}{2}\gamma\{r^6+q^6+15q^2r^2(r^2+q^2)-h^2(r^4+q^4+6r^2q^2)-h^4(r^2+q^2)+h^6\}$$
$$+2H\{r^4+q^4+6r^2q^2-2h^2(r^2+q^2)+h^4\}$$
$$+2\mathrm{K}(r^2+q^2-h^2)-2(r^2+q^2)B^2]\,dt^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2, d),$$

and by comparing the first of these formulæ with the corresponding formulæ of Lagrange, we find, as already observed, that the relation between the constant K and Lagrange's constant C is $\mathrm{K}=C+2Hh^2+B^2-\tfrac{1}{4}\gamma h^4$. And substituting this value of K, the two equations become identical with those of Lagrange.*

The equation $y\dfrac{dz}{dt}-z\dfrac{dy}{dt}=B$, (putting $y=\sqrt{(y^2+z^2)}\cos\phi$, $z=\sqrt{(y^2+z^2)}\sin\phi$) gives at once $(y^2+z^2)\,d\phi=Bdt$, and substituting for y^2+z^2 its value $=\dfrac{\nabla}{4h^2}$, we find

$$d\phi=\frac{4h^2B}{4q^2r^2-(h^2-r^2-q^2)^2}\,dt \quad\ldots\ldots\ldots\ldots (3, d)$$

which is the third of Lagrange's equations.

To complete the solution, the combination of the first and second equations gives

$$r^2q^2(dr\pm dq)^2=[\alpha\{(r\pm q)^3-h^2(r\pm q)\}\pm\beta\{(r\pm q)^3-h^2(r\pm q)\}$$
$$-\tfrac{1}{4}\gamma\{(r\pm q)^6-h^2(r\pm q)^4-h^4(r\pm q)^2+h^6\}$$
$$+H\{(r\pm q)^4-2h^2(r\pm q)^2+h^4\}$$
$$+2\mathrm{K}\{(r\pm q)^2-h^2\}-2B^2(r\pm q)^2]\,dt^2,$$

and thence putting $r+q=s$, $r-q=u$ and writing for shortness

$$S=(\alpha+\beta)(s^3-s^2h)$$
$$-\tfrac{1}{4}\gamma(s^6-h^2s^4-h^4s^2+h^6)$$
$$+H(s^4-2h^2s^2+h^4)$$
$$+2\mathrm{K}(s^2-h^2)-2B^2s^2,$$

* The formulæ referred to are the formulæ (*b*), (*c*), *Mec. Anal.* t. II. page 112 of the second edition and page 97 of the third edition, but there is an inaccuracy in the formulæ (*c*), B^2 ought to be changed into B^2h^2; the error is continued in the subsequent formulæ and besides the constant term $-Ch^2$ is omitted on the right-hand side of the formulæ (*e*) and in the subsequent formulæ, i.e. in the functions of s, u, the term $-B^2$ should be $-B^2h^2-Ch^2$.

and
$$U = (\alpha - \beta)(u^3 - u^2 h)$$
$$- \tfrac{1}{4}\gamma (u^6 - h^2 u^4 - h^4 s^2 + h^6)$$
$$+ H (u^4 - 2h^2 u^2 - h^4)$$
$$+ 2\text{K} (u^2 - h^2) - 2B^2 u^2,$$

we have
$$\tfrac{1}{16} (s^2 - u^2)^2 ds^2 = S dt^2 \ldots\ldots\ldots\ldots\ldots (1, e),$$
$$\tfrac{1}{16} (s^2 - u^2)^2 du^2 = U dt^2 \ldots\ldots\ldots\ldots\ldots (2, e),$$
$$d\phi = -\frac{4h^2 B}{(s^2 - h^2)(u^2 - h^2)} dt \ldots\ldots\ldots\ldots (3, e).$$

and thence finally
$$\frac{ds}{\sqrt{(S)}} = \frac{du}{\sqrt{(U)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1, f),$$
$$dt = \tfrac{1}{4} \left\{ \frac{s^2 ds}{\sqrt{(S)}} - \frac{u^2 du}{\sqrt{(U)}} \right\} \ldots\ldots\ldots\ldots\ldots (2, f),$$
$$d\phi = \frac{Bh^2 ds}{(s^2 - h^2)\sqrt{(S)}} - \frac{Bu^2 du}{(u^2 - h^2)\sqrt{(U)}} \ldots\ldots (3, f),$$

so that the problem is reduced to quadratures, the functions to be integrated involving the square roots of two rational and integral functions of the sixth degree.

2, Stone Buildings,
10*th Nov.*, 1856.

NOTE ON CERTAIN SYSTEMS OF CIRCLES.

By A. Cayley.

It will be convenient to remark at the outset that two concentric circles, the radii of which are in the ratio of $1 : i$ (i being as usual the imaginary unit), are orthotomic,*

* Two concentric circles are, it is well known, conics having a double contact at infinity, and it appears at first sight difficult to reconcile with this, the idea of two particular concentric circles being orthotomic. The explanation is that any two lines through a circular point at infinity may be considered as being at right angles to each other, and therefore any line through a circular point at infinity may be con-

and that the most convenient quasi representation of a circle, the centre of which is real and the radius a pure imaginary quantity, is by means of the concentric orthotomic circle. This being premised consider a circle and a point C. The points of contact of the tangents through C to the circle may be termed the taction points; the points where the chord through C perpendicular to the line joining C with the centre meets the circle, may be termed the section points. It is clear that, for an exterior point, the taction points are real and the section points imaginary, while, for an interior point, the section points are real and the taction points imaginary. A circle having C for its centre and passing through the taction points (in fact the orthotomic circle having C for its centre) is said to be the taction circle. A circle having C for its centre and passing through the section points is said to be the section circle. Of course for an exterior point the taction circle is real and the section circle imaginary; while for an interior point the taction circle is imaginary and the section circle is real. It is proper also to remark that the taction circle and the section circle are concentric orthotomic circles.

Passing now to the case of two systems of orthotomic circles, let MM', NN' be lines at right angles to each other intersecting in R, and let M, M' be real or pure imaginary points on the line MM', equidistant from R. Imagine a system of circles, each of them having its centre on the line NN' and passing through the points M, M' (so that MM' is the radical axis of these circles). There are always on the line NN' two pure imaginary or real points N, N' equidistant from R, such that the circles, each of them having its centre on MM' and passing through the points N, N' (NN' being therefore the radical axis of these circles), are orthotomic to the first mentioned system of circles. Moreover if R be made the centre of a circle passing through M, M', then the concentric orthotomic circle passes through

sidered as being at right angles to itself. The two concentric circles in question have, in fact, at each circular point at infinity a common tangent, but this common tangent must be considered as being at right angles to itself. The paradox disappears entirely upon a homographic deformation of the figure; two lines KL, KM are then defined to be at right angles when joining K with the fixed points I, J, the four lines KL, KM, KI, KJ are a harmonic pencil; but when K coincides with I, then KI is indeterminate and may be taken to be the fourth harmonic of the pencil, i.e. any two lines IL, IM through the point I may be considered as being at right angles.

N, N'; this is in fact only a particular case of the general property.

Suppose now that M, M' being given as the points of intersection of two circles having their centres on NN', it is required to find a circle having for its centre a given point C on NN' and passing through the points M, M'. In the case of M, M' being real, the required circle is obviously given and is always real. But if M, M' are imaginary; then if about any point of MM' as centre a circle be described orthotomic to one of the circles, it will be orthotomic to the other circle, and will meet NN' in the real points N, N'. Now if ρ be the radius of the required circle (i.e. of the circle having C for its centre and passing through the points M, M'), then $\rho^2 = (RC)^2 + (RM)^2 = (RC)^2 - (RN)^2$. Hence if $RC > RN$ or if C lies outside the space NN', ρ^2 is positive or the required circle is real, and the radius is at once constructed from the preceding expression

$$\rho^2 = (RC)^2 - (RN)^2.$$

But if $RC < RN$ or C lies within the space NN', then the required circle is imaginary, but the concentric orthotomic circle is at once constructed from the formula

$$\rho'^2 = RN^2 - RC^2.$$

Suppose now the point C is a centre of similitude of the two circles. The circle having C for its centre and passing through the points M, M' is a taction circle of all the taction circles of the two circles, it may be termed the tactaction circle. The concentric orthotomic of the circle having C for its centre and passing through the points M, M' is a section circle of all the taction circles of the two circles, it may be termed the sectaction circle. Consider first the case where the circles intersect in a pair of real points; here the two centres of similitude are on opposite sides of R; the tactaction circles are both real, the sectaction circles both imaginary. Secondly, the case where the two circles are wholly exterior each to the other, the two centres of similitude lie on the same side of R, viz. the centre of inverse similitude between R and N, the centre of direct similitude beyond N. Hence the tactaction circle corresponding to the centre of direct similitude and the sectaction circle corresponding to the centre of inverse similitude are real, the other tactaction circle and sectaction circle are imaginary. Thirdly, the case where one of the circles is wholly interior to the other; here the two centres of similitude are still on the same side of R, but the centre of direct similitude lies between R and N, and the centre of inverse similitude lies beyond N. Hence the sectaction circle corresponding to the centre of direct similitude and the tactaction circle corresponding to the centre of inverse similitude are real, the other sectaction circle and tactaction circle are imaginary.

To obtain a distinct idea of the methods made use of in Gaultier's "Memoire sur les moyens généraux de construire graphiquement un cercle déterminé par trois conditions," (*Journ. Polyt.* t. IX. p. 124), and in Steiner's "Geometrische Betrachtungen," *Crelle*, t. I. p. 161; it should be remarked that both of these geometers, confining as they do their attention to real circles, do not consider the section circle of an exterior point, or the taction circle of an interior point. The taction circle of an exterior point, or the section circle of an interior point is Gaultier's "Cercle radical," and Steiner's "Potenzkreis," and Steiner also speaks of the radius of this circle as the "Potenz" of its centre in relation to the given circle. The nature of the Cercle radical or Potenzkreis, (i.e. whether it is a taction circle or a section circle) is of course determined as soon as it is known whether the centre is an exterior or an interior point, and Gaultier distinguishes the two cases as the "radical reciproque" and the "radical simple," and in like manner Steiner speaks of the Potenz as being "aüszerlich" or "innerlich." Again, for two circles and for a given centre of similitude Gaultier and Steiner employ the tactaction circle or the sectaction circle, whichever of them is real, Gaultier without giving any distinctive appellation to the circle in question, Steiner calling it the Potenzkreis of the two circles, and in particular the "aüszere Potenzkreis" or the "innere Potenzkreis," according as it has for centre the centre of direct similitude or the centre of inverse similitude.

The preceding properties of circles are of course at once extended to conics passing each of them through the same two points; it is I think worth while to notice what the analogue is of a pair of concentric orthotomic circles. If the fixed points are I, J and if the point corresponding to the centre is K, then the conics are of course conics touching the lines KI, KJ in the points I, J, and one of the conics being given the other is to be determined. It is easily seen that if an arbitrary line through I meets the conics in P, P' and the line KJ in M, then the points I, M, P, P' are a harmonic range, and this condition gives the construction of the second conic; it of course follows that an arbitrary line through J meets the conics in points Q, Q' and the line KI in a point N such that the points J, N, Q, Q' are also a harmonic range. The two conics in question may be termed "inscribed harmonics" each of the other.

Addition. The equation of the tactaction circle, corresponding to the centre of direct (or inverse) similitude, of two given circles, may be found as follows:

Let the equations of the given circles be

$$(x-\alpha)^2+(y-\beta)^2=c^2,$$
$$(x-\alpha')^2+(y-\beta')^2=c'^2,$$

then the coordinates of the centre of direct similitude are

$$\frac{\alpha c'-\alpha' c}{c'-c'},\quad \frac{\beta c'-\beta' c}{c'-c},$$

which are therefore the coordinates of the centre of the tactaction circle; and the equation of this circle is of the form

$$\lambda\,[(x-\alpha)^2+(y-\beta)^2-c^2]+(1-\lambda)\,[(x-\alpha')^2+(y-\beta')^2-c'^2]=0,$$

or expanding and reducing

$$(x^2+y^2)-2\,[\alpha\lambda+\alpha'(1-\lambda)]\,x-2\,[\beta\lambda+\beta'(1-\lambda)]\,y$$
$$+\lambda\,(\alpha^2+\beta^2-c^2)+(1-\lambda)\,(\alpha'^2+\beta'^2-c'^2)=0.$$

We must therefore have

$$\alpha\lambda+\alpha'(1-\lambda)=\frac{\alpha c'-\alpha' c}{c'-c},$$

$$\beta\lambda+\beta'(1-\lambda)=\frac{\beta c'-\beta' c}{c'-c},$$

which are consistent with each other and give

$$\lambda=\frac{c'}{c'-c},\quad 1-\lambda=\frac{-c}{c'-c}.$$

We have then

$$\lambda\,(\alpha^2+\beta^2-c^2)+(1-\lambda)\,(\alpha'^2+\beta'^2-c'^2)$$
$$=\frac{1}{c'-c}\,[c'(\alpha^2+\beta^2)-c(\alpha'^2+\beta'^2)+cc'(c'-c)];$$

and the equation of the tactaction circle is

$$x^2+y^2-2\frac{\alpha c'-\alpha' c}{c'-c}x-2\frac{\beta c'-\beta' c}{c'-c}y$$

$$=\frac{1}{c'-c}[c'(\alpha^2+\beta^2)-c(\alpha'^2+\beta'^2)+cc'(c'-c)],$$

which may also be written

$$\left(x-\frac{\alpha c'-\alpha' c}{c'-c}\right)^2+\left(y-\frac{\beta c'-\beta' c}{c'-c}\right)^2$$

$$=\frac{cc'}{(c'-c)^2}[(\alpha'-\alpha)^2+(\beta'-\beta)^2-(c'-c)^2].$$

We have thus the equation of the tactaction circle corresponding to the centre of direct similitude, and that of the tactaction circle corresponding to the centre of inverse similitude is at once obtained from it by changing the sign of one of the two radii c, c'.

Consider any three circles and combining them in pairs, by what has preceded the equations of the tactaction circles corresponding to the centres of direct similitude will be

$$(c''-c')(x^2+y^2)-2(\alpha' c''-\alpha'' c')x-2(\beta' c''-\beta'' c')y$$
$$+c''(\alpha'^2+\beta'^2)-c'(\alpha''^2+\beta''^2)+c'c''(c''-c')=0,$$

$$(c-c'')(x^2+y^2)-2(\alpha'' c-\alpha c'')x-2(\beta'' c-\beta c'')y$$
$$+c(\alpha''^2+\beta''^2)-c''(\alpha^2+\beta^2)+c''c(c-c'')=0,$$

$$(c'-c)(x^2+y^2)-2(\alpha c'-\alpha' c)x-2(\beta c'-\beta' c)y$$
$$+c'(\alpha^2+\beta^2)-c(\alpha'^2+\beta'^2)+cc'(c'-c)=0,$$

and representing these equations by $U=0$, $U'=0$, $U''=0$, we have identically $cU+c'U'+c''U''=0$, hence the three tactaction circles pass through the same two points, or what is the same thing, have a common radical axis.

No. 6. **May, 1857.**

THE QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

JOHN W. PARKER AND SON, WEST STRAND.

1857.

W. METCALFE, PRINTER, GREEN STREET, CAMBRIDGE.

PRICE FIVE SHILLINGS.

CONTENTS.

NOTICE TO CORRESPONDENTS.

Papers for the Journal and other communications may be addressed to the Editors under cover to Messrs. J. W. PARKER & SON, 445, *West Strand, London, or to* N. M. FERRERS, Esq., *Gonville and Caius College, Cambridge.*

The next Number will appear in October.

THE PLANETARY THEORY.

(Continued from P. 23).

By Rev. PERCIVAL FROST.

To calculate the rate of change of the mean longitude of the disturbed planet.

18. The disturbing function R is a function of r and θ, so far as it depends upon the position of m alone; therefore the change of R due only to the change of the position of m may be found by considering R only as a function of $n_1t+\epsilon_1$, a_1, e_1, ϖ_1, i_1 and Ω_1.

$$\text{Hence } \frac{d(R)}{dt}=\frac{dR}{d\epsilon_1}\frac{d(n_1t+\epsilon_1)}{dt}+\frac{dR}{da_1}\frac{da_1}{dt}+\frac{dR}{de_1}\frac{de_1}{dt}$$
$$+\frac{dR}{d\varpi_1}\frac{d\varpi_1}{dt}+\frac{dR}{di_1}\frac{di_1}{dt}+\frac{dR}{d\Omega_1}\frac{d\Omega_1}{dt},$$

and
$$\frac{d(R)}{dt}=n_1\frac{dR}{d\epsilon_1};$$

therefore
$$\frac{dR}{d\epsilon_1}\cdot\frac{d(n_1t+\epsilon_1)}{dt}=n_1\frac{dR}{d\epsilon_1}-\frac{2n_1a_1^2}{\mu}\frac{dR}{d\epsilon_1}\frac{dR}{da_1}$$
$$-\frac{n_1a_1}{\mu e_1}\{1-e_1^2-\sqrt{(1-e_1^2)}\}\frac{dR}{d\epsilon_1}\frac{dR}{de_1}-\frac{n_1a_1\tan\frac{1}{2}i_1}{\mu\sqrt{(1-e_1^2)}}\frac{dR}{d\varpi_1}\frac{dR}{di_1}$$
$$+\frac{n_1a_1}{\mu\sqrt{(1-e_1^2)}}\left\{\frac{1}{\sin i_1}\frac{dR}{d\Omega_1}+\tan\tfrac{1}{2}i_1\left(\frac{dR}{d\epsilon_1}+\frac{dR}{d\varpi_1}\right)\right\}\frac{dR}{di_1}$$
$$-\frac{n_1a_1}{\mu\sqrt{(1-e_1^2)}\sin i_1}\frac{dR}{di_1}\frac{dR}{d\Omega_1};$$

$$\therefore\frac{d(n_1t+\epsilon_1)}{dt}=n_1-\frac{2n_1a_1^2}{\mu}\frac{dR}{da_1}+\frac{n_1a_1\sqrt{(1-e_1^2)}}{\mu e_1}\{1-\sqrt{(1-e_1^2)}\}\frac{dR}{de_1}$$
$$+\frac{n_1a_1\tan\frac{1}{2}i_1}{\mu\sqrt{(1-e_1^2)}}\frac{dR}{di_1},$$

which is the rate of change of the mean longitude.

19. In the foregoing articles equations have been found which are sufficient for the determination of the elements of the Instantaneous Ellipse at any time, reckoned from a

fixed epoch at which the values of these elements are known: and, from these elements, the position of the planet can be completely determined.

It will be seen that the elements of the instantaneous orbit of the disturbed planet are subject to two distinct species of variation, which are called *Secular Variations* and *Periodic Variations.*

The *Secular Variations* do not depend upon the configuration of the different bodies of the system, but on the relative positions and magnitudes of the orbits themselves. They may either increase indefinitely with the time, or they may be subject to periods of long duration, yet having no reference to the positions of the bodies in their orbits.

The *Periodic Variations* depend only on the positions of the bodies, relatively to each other, or to their nodes or perihelia, and receive the same values whenever the general disposition of the system recurs. Some of these variations are of short period, *i.e.* pass through all their values in one or two revolutions of some of the planets, others are of long duration and depend on the number of revolutions which two or more planets must perform before they again assume the same configuration.

Of these Periodic Variations, some have been distinguished by the name of *Long Inequalities*, and these are not to be confounded with the *Periodic Secular Variations* which are also of long period, but depend, not on the configuration of the planets, but of the orbits.

20. The equations obtained above are capable of solution by successive approximation, in consequence of the smallness of the eccentricities and relative inclinations of the orbits, as exactly as is necessary for comparison of theory with observation; and we should then be in a position to determine the perturbations of the radius vector, longitude and latitude of a disturbed planet, due to the action of the disturbing bodies. But, if it be required to determine only the Periodic Variations of these coordinates of a planet, it is a more simple method to have recourse at once to the equations of motion.

Before completing the account of the solution of the problem under the former aspect, we shall obtain the equations, by means of which the Periodic Variations of the radius vector, and of the longitude and latitude are more directly determined: and, as before, we shall neglect the squares of the disturbing forces.

Equations of motion of the Disturbed Planet.

21. The position of the planet m can be determined by its distance r, the longitude θ on the orbit, and the latitude, or the angular distance from the fixed plane of reference.

Now, if the fixed plane of reference be taken to coincide with the original position of the plane of m's orbit, since the departure from that plane is due to the disturbing forces, the inclination of the plane of the instantaneous orbit is of that order of smallness, and r and θ may be used for the projections on the fixed plane, since they differ by quantities depending on the squares of the inclinations.

Hence for the equation of the radial acceleration,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} + \frac{dR}{dr} \text{ ....(1).}$$

Again, by Vis Viva,

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 = 2\int\left(-\frac{\mu}{r^2}\,dr + \frac{dR}{ds}\,ds\right)$$

in which $\left(\frac{d\zeta}{dt}\right)^2$ being the square of the velocity perpendicular to the plane is of the order of the square of the disturbing force: also $\frac{dR}{ds}\,ds$ is the change of R due to the change of position of m *alone* in the time dt, and since it depends upon m's position only in consequence of being a function of quantities which involve t in the form $nt + \varepsilon$;

therefore, $$\frac{dR}{ds}\,ds = \frac{dR}{d(nt+\varepsilon)}\,d(nt+\varepsilon) = n\,\frac{dR}{d\varepsilon};$$

therefore, $$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{2\mu}{r} + 2n\int\frac{dR}{d\varepsilon}\,dt + C \text{....... (2).}$$

The remaining equation for determining the latitude λ is

$$\frac{d^2z}{dt^2} + \frac{\mu z}{r^3} = \frac{dR}{dz} \text{..................... (3),}$$

where $$z = r\sin\lambda.$$

These three equations are the equations of motion which we shall employ.

To investigate the differential equation for the perturbation of the radius vector.

22. From the equations of motion of the planet m

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} - \frac{dR}{dr} \text{..............} (1),$$

and $$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{2\mu}{r} + 2n\int\frac{dR}{d\varepsilon}\,dt + C \text{......} (2).$$

Multiplying (1) by r, and adding to (2),

$$r\frac{d^2r}{dt^2} + \left(\frac{dr}{dt}\right)^2 = \frac{\mu}{r} + r\frac{dR}{dr} + 2n\int\frac{dR}{d\varepsilon}\,dt + C,$$

or $$\frac{d^2(r^2)}{dt^2} = \frac{2\mu}{r} + 2r\frac{dR}{dr} + 4n\int\frac{dR}{d\varepsilon}\,dt + 2C.$$

If we neglect the terms involving R we have an equation for determining r in the orbit of m when undisturbed by the action of the disturbing forces, and we can obtain the effect of the disturbing forces by writing $r + \delta r$ for r, δr being the part due to the disturbances, and since the squares of the disturbing forces are to be neglected, we may omit all terms but those of the first order in δr, $\frac{d\delta r}{dt}$ and leave the terms involving R unaltered in form.

Thus $$\frac{d^2(r+\delta r)^2}{dt^2} = 2C + \frac{2\mu}{r+\delta r} + 2r\frac{dR}{dr} + 4n\int\frac{dR}{d\varepsilon}\,dt,$$

and $$\frac{d^2r^2}{dt^2} = 2C + \frac{2\mu}{r};$$

therefore, $$\frac{d^2(r\delta r)}{dt^2} + \frac{\mu}{r^3}\,.\,r\delta r = r\frac{dR}{dr} + 2n\int\frac{dR}{d\varepsilon}\,dt,$$

which is the equation for determining δr.

To investigate the perturbation in longitude.

23. Writing $r + \delta r$ and $\theta + \delta\theta$ for r and θ in the equation of Vis Viva,

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{2\mu}{r} + 2n\int\frac{dR}{d\varepsilon}\,dt + C,$$

and neglecting the squares of the disturbing force,

$$\frac{dr}{dt}\frac{d\delta r}{dt} + r\delta r\left(\frac{d\theta}{dt}\right)^2 + r^2\frac{d\theta}{dt}\frac{d\delta\theta}{dt} + \frac{\mu}{r^2}\delta r = n\int\frac{dR}{d\varepsilon}\,dt,$$

and observing that in the undisturbed orbit $r^2 \dfrac{d\theta}{dt} = h$,

and $$r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2 r}{dt^2} + \frac{\mu}{r^2},$$

$$h\frac{d\delta\theta}{dt} + \frac{dr}{dt}\frac{d\delta r}{dt} + \left(\frac{d^2 r}{dt^2} + \frac{2\mu}{r^3}\right)\delta r = n\int \frac{dR}{d\epsilon}\,dt,$$

and $$\frac{2d^2(r\delta r)}{dt^2} + \frac{2\mu}{r^3}\delta r = 2r\frac{dR}{dr} + 4n\int\frac{dR}{d\epsilon}\,dt;$$

$$\therefore\ h\frac{d\delta\theta}{dt} + \frac{d}{dt}\left(\frac{dr}{dt}\delta r\right) - \frac{2d^2(r\delta r)}{dt^2} = -2r\frac{dR}{dr} - 3n\int\frac{dR}{d\epsilon}\,dt;$$

therefore, $h\delta\theta = \dfrac{2d(r\delta r)}{dt} - \dfrac{dr}{dt}\delta r - 2\displaystyle\int r\frac{dR}{dr}\,dt - 3n\iint\frac{dR}{d\epsilon}\,dt^2$,

whence $\delta\theta$ can be found when δr has been calculated.

To investigate the differential equation for the perturbation in latitude.

24. In the equation

$$\frac{d^2 z}{dt^2} + \frac{\mu z}{r^3} = \frac{dR}{dz},$$

if we take for the fixed plane the plane of m's orbit at the commencement of the epoch, and write for z, $r(\lambda + \delta\lambda)$, we obtain the equation

$$\frac{d^2(r\delta\lambda)}{dt^2} + \frac{\mu}{r^3}\cdot r\delta\lambda = \frac{dR}{dz},$$

whence if $\delta\lambda$ be calculated, the latitude may be found relative to any fixed plane inclined at a small angle to the former by adding this value of $\delta\lambda$ to the latitude of m found on supposition that it does not change its plane of motion.

On the developement of R.

25. The first step towards the solution of the equations found above, is the expansion of the disturbing function in ascending powers of the small quantities, the eccentricities and inclinations of the orbits. Now, although it is not possible, within the limits within which it is desirable to confine ourselves, to enter upon the expansion to high orders of the small quantities, still it is necessary, in order to obtain a clear idea of the nature of the approximations, to enter upon some of the points which arise in the developement of R.

We shall therefore endeavour to give some idea of the methods adopted as shortly as possible.

Recurring to the expression for R obtained in Art. 7, and the developements of the coordinates of the planets given in Art. 8, we observe that

$$R = -\frac{m'\{r_1 r_1' \cos(\theta_1' - \theta_1) + zz'\}}{(r_1'^2 + z_1'^2)^{\frac{3}{2}}} + \frac{m'}{\{r_1^2 - 2r_1 r_1' \cos(\theta_1' - \theta_1) + r_1'^2 + (z_1' - z)^2\}^{\frac{1}{2}}},$$

in which expression r_1 and r_1' differ from a_1 and a_1' by quantities depending upon the small numbers e_1, e_1', $\tan i_1$, $\tan i_1'$, and $\theta_1' - \theta_1$ from $n_1't + \epsilon' - n_1t - \epsilon$ by quantities of the same order, and that z, z' are themselves of the order of the inclinations.

The first term can readily be put in the form

$$-\frac{m'r_1}{r_1'^2}\{\cos(\theta_1' - \theta_1) + \tan\lambda \tan\lambda' - \tfrac{1}{2}\tan^2\lambda' + \ldots\},$$

if $z = r_1 \tan\lambda$, and $z' = r_1' \tan\lambda_1'$.

The second term may be written

$$\frac{m'}{\{r_1^2 - 2r_1 r_1' \cos(\theta_1' - \theta_1) + r_1'^2\}^{\frac{1}{2}}} - \frac{m'(r_1' \tan\lambda' - r_1 \tan\lambda)^2}{\{r_1^2 - 2r_1 r_1' \cos(\theta_1' - \theta_1) + r_1'^2\}^{\frac{3}{2}}}.$$

If therefore we write $r_1 = a_1(1 + u)$ and $r_1' = a_1'(1 + u')$,

and $$R' = -\frac{m'a_1}{a_1'^2}\{\cos(\theta_1' - \theta_1) + \tan\lambda \tan\lambda' - \tfrac{1}{2}\tan^2\lambda' + \ldots\}$$

$$+ \frac{m'}{\{a_1^2 - 2a_1 a_1' \cos(\theta_1' - \theta_1) + a_1'^2\}^{\frac{1}{2}}} - \frac{m'(a_1' \tan\lambda' - a_1 \tan\lambda)^2}{\{a_1^2 - 2a_1 a_1' \cos(\theta_1' - \theta_1) + a_1'^2\}^{\frac{3}{2}}},$$

we obtain R by Taylor's theorem in the form

$$R' + \frac{dR'}{da_1} a_1 u + \frac{dR'}{da_1'} a_1' u' + \ldots.$$

Expansion of $(a^2 - 2aa' \cos\phi + a'^2)^{-\frac{1}{2}}$ *in a series of simple cosines.*

26. Let

$$(a^2 - 2aa' \cos\phi + a'^2)^{-\frac{1}{2}} = \tfrac{1}{2}C_0 + C_1 \cos\phi + C_2 \cos 2\phi + \ldots \quad (1).$$

(1) To calculate the values of C_0, C_1.

Since $a^2 - 2aa' \cos\phi + a'^2 = (a+a')^2 - 2aa'(1-\cos\phi)$.

Let $c^2 = \frac{4aa'}{(a+a')^2}$, and for ϕ write $\pi - 2\psi$; therefore

$$\frac{1}{(a+a')\sqrt{(1-c^2\sin^2\psi)}} = \tfrac{1}{2}C_0 - C_1\cos 2\psi + C_2\cos 4\psi - \ldots;$$

therefore, integrating from $\psi = 0$ to $\psi = \frac{\pi}{2}$,

$$\frac{\pi}{4}C_0 = \frac{1}{a+a'}\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{(1-c^2\sin^2\psi)}}.$$

Again multiplying both sides by $\cos 2\psi$ and integrating as before

$$-\frac{\pi}{4}C_1 = \frac{1}{a+a'}\int_0^{\frac{\pi}{2}} \frac{(1-2\sin^2\psi)\,d\psi}{\sqrt{(1-c^2\sin^2\psi)}}.$$

These definite integrals are included in the form

$$\int_0^{\frac{\pi}{2}} \frac{(a+b\sin^2\psi)\,d\psi}{\sqrt{(1-c^2\sin^2\psi)}},$$

which is numerically calculated by reducing it to another integral of the same form in which c is less, and so on by successive reductions until the c becomes insensible. This may be effected as follows:

If
$$\sin(2\psi - \psi_1) = c_1 \sin\psi_1,$$
$$\cos(2\psi - \psi_1)(2d\psi - d\psi_1) = c_1\cos\psi_1\, d\psi_1;$$

therefore
$$\frac{d\psi_1}{\cos(2\psi-\psi_1)} = \frac{2d\psi}{\cos(2\psi-\psi_1) + c_1\cos\psi_1}.$$

Now
$$\frac{\cos\psi_1}{\cos 2\psi + c_1} = \frac{\sin\psi_1}{\sin 2\psi} = \frac{1}{\sqrt{(1+2c_1\cos 2\psi + c_1^2)}}$$
$$= \frac{\cos(2\psi-\psi_1) + c_1\cos\psi_1}{1+2c_1\cos 2\psi + c_1^2};$$

therefore

$$\frac{d\psi_1}{\sqrt{(1-c_1^2\sin^2\psi_1)}} = \frac{2d\psi}{\sqrt{(1+2c_1\cos 2\psi + c_1^2)}} = \frac{2d\psi}{(1+c_1)\sqrt{(1-c^2\sin^2\psi)}},$$

if
$$c^2 = \frac{4c_1}{(1+c_1)^2},$$

and $\sin^2\psi = \frac{1}{2}(1-\cos 2\psi) = \frac{1}{2}\{1-\cos(2\psi - \psi_1 + \psi_1)\}$

$$= \tfrac{1}{2}\{1 + c_1 \sin^2\psi_1 - \cos\psi_1 \sqrt{(1-c_1^2\sin^2\psi_1)}\},$$

and if $\psi = 0, \quad \psi_1 = 0,$

$$\psi = \frac{\pi}{2}, \quad \psi_1 = \pi;$$

therefore

$$\int_0^{\frac{\pi}{2}} \frac{(a+b\sin^2\psi)d\psi}{\sqrt{(1-c^2\sin^2\psi)}} = \int_0^{\pi} \frac{1+c_1}{2}\left\{\frac{a+\frac{b}{2}+\frac{bc_1}{2}\sin^2\psi_1}{\sqrt{(1-c_1^2\sin^2\psi_1)}} - \frac{b}{2}\cos\psi_1\right\}d\psi_1$$

$$= (1+c_1)\int_0^{\frac{\pi}{2}} \frac{a_1 + b_1\sin^2\psi_1}{\sqrt{(1-c_1^2\sin^2\psi)}}.d\psi_1,$$

where $a_1 = a + \frac{b}{2}$ and $b_1 = \frac{bc_1}{2},$

and $$c_1 = \frac{1-\sqrt{(1-c^2)}}{1+\sqrt{(1-c^2)}};$$

therefore $$\frac{c_1}{c} = \frac{c}{\{1+\sqrt{(1-c^2)}\}^2},$$

and $c < 1$; therefore $c_1 < c$,

whence, if $c_2 c_3 \ldots$ be obtained successively as c_1 from c, c_r at length becomes insensible, and therefore b_r,

and $$\int_0^{\frac{\pi}{2}} \frac{a_r + b_r \sin^2\psi_r}{\sqrt{(1-c_r^2\sin^2\psi_r)}}\,d\psi_r = \frac{\pi}{2}\,a_r;$$

therefore

$$\int_0^{\frac{\pi}{2}} \frac{a+b\sin^2\psi}{\sqrt{(1-c^2\sin^2\psi)}}\,d\psi = \frac{\pi}{2}(1+c_1)(1+c_2)\ldots\left\{a + \frac{b}{2}\left(1+\frac{c_1}{2}+\ldots\right)\right\};$$

therefore

$$\tfrac{1}{2}C_0 = \frac{1}{a+a'}(1+c_1)(1+c_2)(1+c_3)\ldots$$

$$C_1 = -\frac{2}{a+a'}(1+c_1)(1+c_2)(1+c_3)\ldots\left\{1-1-\frac{c_1}{2}-\frac{c_1c_2}{2^2}\ldots\right\}$$

$$= C_0\left(\frac{c_1}{2}+\frac{c_1c_2}{2^2}+\frac{c_1c_2c_3}{2^3}+\ldots\right),$$

whence C_0 and C_1 can be calculated to any degree of accuracy.

The numerical calculation is rendered very simple by assuming

$$c_0 = \sin\alpha,$$

$$c_1 = \tan^2\frac{\alpha}{2} = \sin\alpha',$$

$$c_2 = \tan^2\frac{\alpha'}{2} = \sin\alpha'',$$

....

$$C_0 = \frac{2}{a+a'}\sec^2\frac{\alpha}{2}.\sec^2\frac{\alpha'}{2}\ldots,$$

$$C_1 = C_0\left(\tfrac{1}{2}\sin\alpha' + \frac{1}{2^2}\sin\alpha'\sin\alpha'' + \ldots\right).$$

(2) To calculate C_2, C_3, ...

Differentiating the equation (1) with respect to ϕ,

$$\frac{aa'\sin\phi}{(a^2 - 2aa'\cos\phi + a'^2)^{\frac{3}{2}}} = C_1\sin\phi + 2C_2\sin 2\phi + \ldots;$$

therefore, multiplying by $a^2 - 2aa'\cos\phi + a'^2$,

$$aa'\sin\phi\,(\tfrac{1}{2}C_0 + C_1\cos\phi + C_2\cos 2\phi + \ldots)$$
$$= (a^2 - 2aa'\cos\phi + a'^2)(C_1\sin\phi + 2C_2\sin 2\phi + \ldots);$$

therefore $aa'\{C_0\sin\phi + C_1\sin 2\phi + C_2(\sin 3\phi - \sin\phi) + \ldots\}$

$$= 2(a^2 + a'^2)(C_1\sin\phi + 2C_2\sin 2\phi + \ldots)$$
$$- 2aa'\{C_1\sin 2\phi + 2C_2(\sin 3\phi + \sin\phi) + \ldots\},$$

and equating the coefficients of $\sin k\phi$

$$aa'(C_{k-1} - C_{k+1}) = 2(a^2 + a'^2)kC_k - 2aa'\{(k-1)C_{k-1} + (k+1)C_{k+1}\};$$

therefore $(2k+1)C_{k+1} + (2k-1)C_{k-1} - 2k.\dfrac{a^2 + a'^2}{aa'}C_k = 0,$

whence the complete series can be determined.

Expansion of $(a^2 - 2aa'\cos\phi + a'^2)^{-\frac{3}{2}}$.

27. Let $(a^2 - 2aa'\cos\phi + a'^2)^{-\frac{3}{2}} = \tfrac{1}{2}D_0 + D_1\cos\phi + D_2\cos 2\phi + \ldots$ (2);

therefore $\tfrac{1}{2}C_0 + C_1\cos\phi + C_2\cos 2\phi + \ldots$

$$= (a^2 + a'^2 - 2aa'\cos\phi)(\tfrac{1}{2}D_0 + D_1\cos\phi + D_2\cos 2\phi + \ldots)$$
$$= (a^2 + a'^2)(\tfrac{1}{2}D_0 + D_1\cos\phi + D_2\cos 2\phi + \ldots)$$
$$- aa'\{D_0\cos\phi + D_1(1 + \cos 2\phi) + D_2(\cos 3\phi + \cos\phi) + \ldots\};$$

therefore equating coefficients of $\cos k\phi$,

$$C_k = (a^2 + a'^2) D_k - aa' (D_{k-1} + D_{k+1}),$$

and proceeding as in the last article,

$$(2k-1) D_{k+1} + (2k+1) D_{k-1} - 2k \frac{a^2 + a'^2}{aa'} D_k = 0;$$

therefore $(2k-1) C_k = (2k-1) \{(a^2 + a'^2) D_k - aa' D_{k-1}\}$

$$+ aa' (2k+1) D_{k-1} - 2k (a^2 + a'^2) D_k$$

$$= -(a^2 + a'^2) D_k + 2aa' D_{k-1};$$

therefore $(2k+1) C_{k+1} = -(a^2 + a'^2) D_{k+1} + 2aa' D_k;$

therefore $(4k^2 - 1) C_{k+1} = (a^2 + a'^2) \left\{(2k+1) D_{k-1} - 2k \frac{a^2 + a'^2}{aa'} D_k\right\}$

$$+ 2aa' (2k-1) D_k$$

$$= -\left\{2k . \frac{(a^2 + a'^2)^2}{aa'} - (2k-1)\, 2aa'\right\} D_k$$

$$+ (2k+1) (a^2 + a'^2) D_{k-1};$$

therefore $(4k^2 - 1) \{(a^2 + a'^2) C_k - 2aa' C_{k+1}\}$

$$= \{4k (a^2 + a'^2)^2 - (2k-1)\, 4a^2 a'^2 - (2k+1) (a^2 + a'^2)^2\} D_k$$

$$= (2k-1) (a^2 - a'^2)^2 D_k;$$

therefore $\quad D_k = \dfrac{(2k+1) \{(a^2 + a'^2) C_k - 2aa' C_{k+1}\}}{(a^2 - a'^2)^2},$

whence D_0, D_1, &c. can be completely determined.

To calculate the differential coefficients of C_k *and* D_k *with respect to* a, a'.

28. Differentiating equation (1), of Art. 26, with respect to a

$$-\frac{a - a' \cos\phi}{(a^2 - 2aa' \cos\phi + a'^2)^{\frac{3}{2}}} = \frac{1}{2} \frac{dC_0}{da} + \frac{dC_1}{da} \cos\phi + \ldots$$

$$= -\frac{1}{2a} . \frac{a^2 - 2aa' \cos\phi + a'^2 + a^2 - a'^2}{(a^2 - 2aa' \cos\phi + a'^2)^{\frac{3}{2}}}$$

$$= -\frac{1}{2a} (\tfrac{1}{2} C_0 + C_1 \cos\phi + \ldots)$$

$$- \frac{a^2 - a'^2}{2a} (\tfrac{1}{2} D_0 + D_1 \cos\phi + \ldots);$$

therefore $$\frac{dC_k}{da} = -\frac{1}{2a}C_k - \frac{a^2-a'^2}{2a}D_k$$

$$= -\frac{(2k+1)\{(a^2+a'^2)C_k - 2aa'C_{k+1}\}}{2a(a^2-a'^2)} \dots (2)$$

$$= \frac{(k+1)a^2+ka'^2}{a(a'^2-a^2)}C_k - \frac{(2k+1)a'}{a'^2-a^2}C_{k+1}.$$

Also C_k being a homogeneous function in a and a' of the order -1,

$$a'\frac{dC_k}{da'} + a\frac{dC_k}{da} = -C_k,$$

$\dfrac{d^2C_k}{da^2}$ can be determined by differentiating (2),

and $$a'\frac{d^2C_k}{da'da} + a\frac{d^2C_k}{da^2} = -2\frac{dC_k}{da},$$

$$a'\frac{d^2C_k}{da'^2} + a\frac{d^2C_k}{da'da} = -2\frac{dC_k}{da'}.$$

Thus the differential coefficients can all be determined.

29. The differential coefficients of C_k can be found in a very simple form in terms of D_k, &c.

Differentiating (1) with respect to a

$$\frac{1}{2}\frac{dC_0}{da} + \frac{dC_1}{da}\cos\phi + \dots\dots = -\frac{a-a'\cos\phi}{(a^2-2aa'\cos\phi+a'^2)^{\frac{3}{2}}}$$

$$= -(a-a'\cos\phi)(\tfrac{1}{2}D_0 + D_1\cos\phi + \dots\dots)$$

$$= -a(\tfrac{1}{2}D_0 + D_1\cos\phi + \dots) + \frac{a'}{2}\{D_0\cos\phi + D_1(1+\cos 2\phi) + \dots\};$$

therefore $$\frac{dC_k}{da} = -aD_k + \frac{a'}{2}(D_{k-1} + D_{k+1}),$$

and $$\frac{dC_k}{da'} = -a'D_k + \frac{a}{2}(D_{k-1} + D_{k+1}),$$

$$\frac{d^2C_k}{da^2} = -D_k - a\frac{dD_k}{da} + \frac{a'}{2}\left(\frac{dD_{k-1}}{da} + \frac{dD_{k+1}}{da}\right).$$

Differentiating (2), of Art. 27, with respect to a and ϕ,

$$\frac{1}{2}\frac{dD_0}{da} + \frac{dD_1}{da}\cos\phi + \dots\dots = -3(a^2-2aa'\cos\phi+a'^2)^{-\frac{5}{2}}(a+a'\cos\phi),$$

$$D_1\sin\phi + 2D_2\cos 2\phi + \dots\dots = 3(a^2-2aa'\cos\phi+a'^2)^{-\frac{5}{2}}aa'\sin\phi;$$

therefore $a\left(\frac{1}{2}\frac{dD_0}{da}+\frac{dD'}{da}\cos\phi+\ldots\ldots\right)(a-a'\cos\phi)$

$$-a'(D_1\sin\phi+2D_2\sin 2\phi+\ldots\ldots)\sin\phi$$

$$=-3a(a^2-2aa'\cos\phi+a'^2)^{-\frac{5}{2}}\{(a-a'\cos\phi)^2+a'^2\sin^2\phi\}$$

$$=-3a(a^2-2aa'\cos\phi+a'^2)^{-\frac{3}{2}}=-3a(\tfrac{1}{2}D_0+D_1\cos\phi+\ldots\ldots);$$

therefore $a\left\{a\frac{dD_k}{da}-\frac{1}{2}a'\left(\frac{dD_{k-1}}{da}+\frac{dD_{k+1}}{da}\right)\right\}$

$$-\tfrac{1}{2}a'\{(k+1)D_{k+1}-(k-1)D_{k-1}\}$$
$$=-3aD_k;$$

therefore $\frac{d^2C_k}{da^2}=-D_k-\frac{a'}{2a}\{(k+1)D_{k+1}-(k-1)D_{k-1}\}+3D_k$

$$=2D_k-\frac{a'}{2a}\{(k+1)D_{k+1}-(k-1)D_{k-1}.$$

And $\frac{dC_k}{da}$ being a homogeneous function of a and a' of degree -2,

$$a\frac{d^2C_k}{da^2}+a'\frac{d^2C_k}{da\,da'}=-2\frac{dC_k}{da};$$

therefore $\quad a'\frac{d^2C_k}{da\,da'}=2aD_k-a'(D_{k-1}+D_{k+1})$

$$+\tfrac{1}{2}a'\{(k+1)D_{k+1}-(k-1)D_{k-1}\}-2aD_k;$$

therefore $\quad \frac{d^2C_k}{da\,da'}=\frac{1}{2}(k-1)D_{k+1}-\frac{1}{2}(k+1)D_{k-1},$

whence $\quad \frac{d^2C_0}{da\,da'}=-D_1$ and $\frac{d^2C_1}{da\,da'}=-D_0$

results which will be useful hereafter.

We have now shewn how to obtain R' in the form

$$m'\Big[\tfrac{1}{2}C_0+\left(C_1+\frac{a_1}{a_1'^2}\right)\cos(\theta_1'-\theta_1)+C_2\cos 2(\theta_1'-\theta_1)+\ldots\ldots$$
$$-\frac{a_1}{a_1'^2}(\tan\lambda \text{ to } \lambda'-\tfrac{3}{2}\tan^2\lambda'+\ldots\ldots)$$
$$-(a_1'\tan\lambda'-a_1\tan\lambda)^2\{\tfrac{1}{2}D_0+D_1\cos(\theta_1'-\theta_1)+D_2\cos 2(\theta_1'-\theta_1)\ldots\}\Big],$$

and $\frac{dR'}{da_1}$, $\frac{dR'}{da_1'}$, &c., can be determined immediately,

and $\quad R=R'+\frac{dR'}{da_1}a_1u+\frac{dR'}{da_1'}a_1'u'+\frac{1}{2}\frac{d^2R'}{da_1^2}a_1^2u^2+\ldots\ldots$

In order to expand R in a series of simple cosines the above arguments are put into the form $pnt - qn't + \alpha$, we observe that

$$\theta_1' - \theta = n_1' + \epsilon_1' - n_1 t - \epsilon_1 + v' - v,$$

$$\tan\lambda = \tan i_1 \sin(\theta_1 - \Omega_1),$$

and
$$\tan\lambda' = \tan i_1' \sin(\theta_1' - \Omega_1');$$

where
$$v' - v = 2e_1' \sin(n_1't + \epsilon_1' - \varpi_1') - 2e \sin(n_1 t + \epsilon_1 - \varpi_1)$$
$$+ \frac{5e_1'^2}{4} \sin 2(n_1't + \epsilon_1' - \varpi_1') - \frac{5e_1^2}{4} \sin 2(n_1 t + \epsilon_1 - \varpi_1)$$
$$- \sin^2\frac{i_1'}{2} \sin 2(n_1't + \epsilon_1' - \Omega_1') + \sin^2\frac{i}{2} \sin 2(n_1 t + \epsilon_1 - \Omega_1)$$
$$+ \ldots\ldots\ldots\ldots\ldots$$

whence
$$\cos k(\theta_1' - \theta_1) = \cos k(n_1't + \epsilon_1' - n_1 t - \epsilon_1)$$
$$- k \sin k(n_1' + \epsilon_1' - n_1 t - \epsilon_1) \,.\, (v' - v)$$
$$- \tfrac{1}{2}k^2 \cos k\,(n_1't + \epsilon_1' - n_1 t - \epsilon_1)\,(v' - v)^2 + \ldots\ldots$$

and the expansion can be effected by the ordinary trigonometrical formulæ for transforming powers and products of sines and cosines into series of simple cosines.

To calculate the constant part of R when expanded in a series of simple cosines.

30. The expansion of R as far as the second order of small quantities is

$$m'\left\{\tfrac{1}{2} C_0 + \left(C_1 - \frac{a_1}{a_1'^2}\right)\cos\phi + C_2 \cos 2\phi + \ldots\ldots\right\}$$
$$+ \left\{\tfrac{1}{2}\frac{dC_0}{da_1} + \left(\frac{dC_1}{da_1} - \frac{1}{a_1'^2}\right)\cos\phi + \frac{dC_2}{da_1}\cos 2\phi + \ldots\ldots\right\} a_1 u$$
$$+ \left\{\tfrac{1}{2}\frac{dC_0}{da_1'} + \left(\frac{dC_1}{da_1'} + \frac{2a_1}{a_1'^3}\right)\cos\phi + \frac{dC_2}{da_1'}\cos 2\phi + \ldots\ldots\right\} a_1'u'$$
$$+ \left(\tfrac{1}{2}\frac{d^2C_0}{da_1^2} + \frac{d^2C_1}{da_1^2}\cos\phi + \ldots\ldots\right)\tfrac{1}{2}a_1^2u^2$$
$$+ \left\{\tfrac{1}{2}\frac{d^2C_0}{da_1 da_1'} + \left(\frac{d^2C_1}{da_1 da_1'} + \frac{2}{a_1'^3}\right)\cos\phi + \ldots\ldots\right\} a_1 a_1' u u'$$
$$+ \ldots\ldots\ldots\ldots$$
$$- \tfrac{1}{2}\left(\tfrac{1}{2}D_0 + D_1\cos\phi + \ldots\ldots\right)(a_1'\tan\lambda' - a_1\tan\lambda)^2$$
$$- \frac{a_1}{a_1'^2}\left(\tan\lambda\tan\lambda' - \tfrac{3}{2}\tan^2\lambda'\right),$$

where $u = -e_1 \cos(n_1 t + \epsilon_1 \varpi_1) + \frac{e_1^2}{2}\{1 - \cos 2(n_1 t + \epsilon_1 - \varpi_1)\} + \ldots$

$$- \tfrac{1}{4}\tan^2 i_1 \{1 - \cos 2(n_1 t + \epsilon_1 - \Omega_1)\} + \ldots\ldots$$

$$\phi = n_1' t + \epsilon_1' - n_1 t - \epsilon + v' - v,$$

$$v = 2e_1 \sin(n_1 t + \epsilon_1 - \varpi_1) + \frac{5e_1^2}{4}\sin 2(n_1 t + \epsilon_1 - \varpi_1) + \ldots\ldots$$

and $\qquad \tan\lambda = \tan i_1 \sin(n_1 t + \epsilon_1 - \Omega_1),$

and similar expressions for u', v', and $\tan\lambda'$.

We must now examine the terms in order, which give rise to terms independent of the time, to the second degree of small quantities; and, for the sake of shortness, we will write I for $n_1' t + \epsilon_1' - n_1 t - \epsilon_1$.

(1) $\cos\phi = \cos(I + v' - v) = \cos I\{1 - \frac{1}{2}(v' - v)^2\} - \sin I(v' - v),$

$v'v$ gives rise to the term

$$4ee' \sin(n_1' t + \epsilon_1' - \varpi_1') \sin(n_1 t + \epsilon_1 - \varpi_1),$$

and therefore to $\qquad 2e_1 e_1' \cos(I - \varpi_1' + \varpi_1)$;

$\therefore$ $\cos\phi$ gives rise to the term $2e_1 e_1' \cos I \cos(I - \varpi_1' + \varpi_1)$, and therefore contains a constant term $e_1 e_1' \cos(\varpi_1' - \varpi_1)$.

(2) u contains the constant term $\frac{1}{2}e_1^2 - \frac{1}{4}\tan^2 i_1$,

and $\quad u'$ .... $\frac{1}{2}e_1'^2 - \frac{1}{4}\tan^2 i_1'$.

(3) $u \cos\phi$ contains $\quad -uv' \sin I$,

uv' contains $\;-2e_1 e_1' \sin(n_1' t + \epsilon_1' - \varpi_1') \cos(n_1 t + \epsilon_1 - \varpi_1)$,

and therefore $\qquad e_1 e_1' \sin(I - \varpi_1' + \varpi_1)$;

therefore $u \cos\phi$ contains a constant part $-\frac{1}{2}e_1 e_1' \cos(\varpi_1' - \varpi_1)$,

$\qquad u' \cos\phi$ .... $-\frac{1}{2}e_1 e_1' \cos(\varpi_1' - \varpi_1)$.

(4) u^2 and u'^2 contains $\frac{1}{2}e_1^2$ and $\frac{1}{2}e_1'^2$ respectively.

(5) uu' contains $e_1 e_1' \cos(n_1' t + e_1' - \varpi_1') \cos(n_1 t + \epsilon_1 - \varpi_1)$,

and therefore $\qquad \frac{1}{2}e_1 e_1' \cos(I - \varpi_1' + \varpi_1)$;

therefore $uu' \cos\phi$ contains a constant part $\frac{1}{4}e_1 e_1' \cos(\varpi_1' - \varpi_1)$.

(6) $2\tan\lambda \tan\lambda'$ contains

$$\tan i_1 \tan i_1' \sin(n_1' + \epsilon_1' - \Omega_1') \sin(n_1 t + \epsilon_1 - \Omega_1),$$

and therefore $\quad \frac{1}{2}\tan i_1 \tan i_1' \cos(I - \Omega_1' + \Omega_1)$;

therefore $2\tan\lambda\tan\lambda'\cos\phi$ contains a constant part

$$\tfrac{1}{2}\tan i_1 \tan i_1' \cos(\Omega_1' - \Omega_1).$$

(7) $\tan^2\lambda$ and $\tan^2\lambda'$ contain $\frac{1}{2}\tan^2 i_1$ and $\frac{1}{2}\tan^2 i_1'$ respectively.

If F be the constant part of R, we obtain, collecting all the terms,

$$\begin{aligned}
F = m' \Big\{ & \tfrac{1}{2}C_0 + \left(C_1 - \frac{a_1}{a_1'^2}\right) \tfrac{1}{2}e_1e_1' \cos(\varpi_1' - \varpi_1) \\
& + \left(\frac{dC_1}{da_1'} - \frac{1}{a_1'^2}\right) \tfrac{1}{2}a_1e_1e_1' \cos(\varpi_1' - \varpi_1) \\
& + \left(\frac{dC_1}{da_1'} + \frac{2a_1}{a_1'^3}\right) \tfrac{1}{2}a_1'e_1e_1' \cos(\varpi_1' - \varpi_1) \\
& + \tfrac{1}{4}\left(a_1 \frac{dC_0}{da_1} e_1^2 + a_1' \frac{dC_0}{da_1'} e_1'^2\right) \\
& + \tfrac{1}{8}\left(a_1^2 \frac{d^2C_0}{da_1^2} e_1^2 + a_1'^2 \frac{d^2C_0}{da_1} e_1'^2\right) \\
& + \tfrac{1}{4}a_1a_1' \left(\frac{d^2C_1}{da_1\,da_1'} + \frac{2}{a'^3}\right) e_1e_1' \cos(\varpi_1' - \varpi_1) \\
& + \tfrac{1}{4}a_1a_1'D_1 \tan i_1 \tan i_1' \cos(\Omega_1' - \Omega_1) \\
& - \tfrac{1}{8}\left(a_1 \frac{dC_0}{da_1} \tan^2 i_1 + a_1' \frac{dC_0}{da_1'} \tan^2 i_1'\right) \\
& - \tfrac{1}{8} D_0 (a_1'^2 \tan^2 i_1' + a_1^2 \tan^2 i_1) \Big\}
\end{aligned}$$

$$\begin{aligned}
= m' \Big\{ & \tfrac{1}{2}C_0 + \tfrac{1}{4}\left(a_1 \frac{dC_0}{da_1} + \tfrac{1}{2}a_1^2 \frac{d^2C_0}{da_1^2}\right) e_1^2 + \tfrac{1}{4}\left(a_1' \frac{dC_0}{da_1'} + \tfrac{1}{2}a_1'^2 \frac{dC_0}{da_1'^2}\right) e_1'^2 \\
& + \tfrac{1}{4}\left(4C_1 + 2a_1 \frac{dC_1}{da_1} + 2a_1' \frac{dC_1}{da_1'} + a_1a_1' \frac{d^2C_1}{da_1\,da_1'}\right) e_1e_1' \cos(\varpi_1' - \varpi_1) \\
& - \tfrac{1}{8}\left(a_1^2D_0 + a_1 \frac{dC_0}{da_1}\right) \tan^2 i_1 - \tfrac{1}{8}\left(a_1'^2D_0 + a_1' \frac{dC_0}{da_1'}\right) \tan^2 i_1' \\
& + \tfrac{1}{4}a_1a_1'D_1 \tan i_1 \tan i_1' \cos(\Omega_1' - \Omega_1) \Big\}.
\end{aligned}$$

Now $\frac{1}{2}C_0 + C_1\cos\phi + \ldots\ldots = (a_1^2 - 2a_1a_1'\cos\phi + a_1'^2)^{-\frac{1}{2}}$;

$$\therefore\ \tfrac{1}{2}\frac{dC_0}{da_1} + \ldots = (\tfrac{1}{2}D_0 + D_1\cos\phi + \ldots)(a_1'\cos\phi - a_1) \ldots (1);$$

therefore $$\tfrac{1}{2}\frac{dC_0}{da_1} = -\tfrac{1}{2}a_1D_0 + \tfrac{1}{2}a_1'D_1;$$

therefore $$a_1^2 D_0 + a_1 \frac{dC_0}{da_1} = a_1 a_1' D_1,$$

and similarly, $$a_1'^2 D_0 + a_1' \frac{dC_0}{da_1'} = a_1 a_1' D_1.$$

Again, $\frac{dC_0}{da_1}$ being a homogeneous function of degree -2,

$$a_1 \frac{d^2 C_0}{da_1^2} + a_1' \frac{d^2 C_0}{da_1\, da_1'} = -2\frac{dC_0}{da_1};$$

therefore, $$a_1 \frac{dC_0}{da_1} + \tfrac{1}{2} a_1^2 \frac{d^2 C_0}{da_1^2} = -\tfrac{1}{2} a_1 a_1' \frac{d^2 C_0}{da_1\, da_1'}$$

$$= a_1' \frac{dC_0}{da_1'} + \tfrac{1}{2} a_1'^2 \frac{d^2 C_0}{da_1'^2} \text{ similarly;}$$

therefore the coefficient of δe_1^2 and $\delta e_1'^2$ each $= \tfrac{1}{8} a_1 a_1' D_1$.

Again, the coefficient of

$$\tfrac{1}{2} e_1 e_1' \cos(\varpi_1' - \varpi_1) = 2c_1 + a_1 a_1' \frac{d^2 C_1}{da_1\, da_1'},$$

and $C_1 \sin\phi + \ldots\ldots = (\tfrac{1}{2} D_0 + D_1 \cos\phi + \ldots\ldots)\, a_1 a_1' \sin\phi$;

therefore $$C_1 = \tfrac{1}{2}(D_0 - D_2)\, a_1 a_1',$$

and $$\frac{d^2 C_1}{da_1\, da_1'} = -D_0. \text{ See Art. 31;}$$

therefore $$2C_1 + a_1 a_1' \frac{d^2 C_1}{da_1\, da_1'} = -a_1 a_1' D_2.$$

Hence

$$F = m' [\tfrac{1}{2} C_0 + \tfrac{1}{8} a_1 a_1' \{D_1 (e_1^2 + e_1'^2) - 2D_2 e_1 e_1' \cos(\varpi_1' - \varpi_1)\}$$
$$- \tfrac{1}{8} a_1 a_1' D_1 \{\tan^2 i_1 + \tan^2 i_1' - 2 \tan i_1 \tan i_1' \cos(\Omega_1' - \Omega_1)\}];$$

in which all the coefficients have been investigated.

On the order of the terms in the complete development of R *in a series of simple cosines.*

31. The last step which we shall give in the development of R is that in which the order of the terms of the form

$$P \cos\{p(n_1' t + \epsilon_1') - q(n_1 t + \epsilon_1) + \alpha\},$$

and $$Q \cos\{p(n_1' t + \epsilon_1') + q(n_1 t + \epsilon_1) + \beta\}$$

is determined, in which it will be shewn that P is of the order $p \sim q$, and Q of the order $p + q$.

The discovery of the relation which the coefficient of any term bears to the argument is essential, because it enables us to select *a priori* those terms, among the infinite number into which R is developed, which alone can give rise to disturbances which will be sensible.

For example, one term in the expression for $\delta\theta$ given in Art. 23, is $-3n_1\iint\frac{dR}{d\varepsilon_1}dt^2$. Hence, a term

$$P\cos\{p(n_1't+\varepsilon_1')-q(n_1t+\varepsilon_1)+\alpha\}$$

in the development of R would give rise to a term

$$-\frac{3n_1qP}{(pn_1'-qn_1)^2}\cos\{p(n_1't+\varepsilon_1')-q(n_1t+\varepsilon_1)+\alpha\}.$$

If therefore the ratio of the mean angular velocities of m' and m, viz., $n_1':n_1$ be very nearly the ratio of two integers q and p, which are not very large, $pn_1'-qn_1$ will be very small, and although P is of the order $p\sim q$ and may be extremely small, the term to which it gives rise in $\delta\theta$ may be large enough to become sensible.

Thus, in the case of Jupiter and Saturn, the ratio is 2 : 5, and in that of Venus and the Earth 13 : 8, so nearly that the terms in R which are of the order 3 and 5 respectively, become sensible.

Order of the term P $\cos(\mathrm{pn_1't-qn_1t}+\alpha)$.

32. To determine the order of the terms, we must refer to the expression for R given in the Art. 30.

(1) In the expansions of u, u', v, v', $\tan\lambda$, $\tan\lambda'$ we observe that the following remarkable law holds: that the order of any term whose argument is $pn_1t+\alpha$, or $pn_1't+\alpha'$ is p the multiplier of n_1t or $n_1't$.

(2) The same law obtains in the expansions of the powers and products of powers of u, v, $\tan\lambda$.

For, suppose two series in which this law holds to be multiplied together, of which the general terms are

$$L\cos(lnt+\alpha) \text{ and } M\cos(mnt+\beta),$$

the general term of the product is

$$\tfrac{1}{2}LM[\cos\{(l-m)nt+\alpha-\beta\}+\cos\{(l+m)nt+\alpha+\beta\}].$$

hence a term of the form $P\cos(pnt+\gamma)$ is the sum of all the terms for which $l+m$ or $l\sim m=p$, the order is therefore that of LM, and the principal part of P is of the order of the smallest value which $l+m$ can assume consistent with those conditions, and since $l \sim m = p+2m$, or $p+2l$, and $l+m=p$, p is the order of the term in the product; therefore the law holds for the product of two such series, and since powers and products of powers of u, v, $\tan\lambda$ can be formed by the continual multiplication of such series, the law holds for these powers and products, and the same is true for u', v', $\tan\lambda'$.

(3) In the expansion of products of powers of u, u' as u^ρ, u'^σ the order of the coefficient of a term whose argument is $(pn_1' \pm qn_1)t+\alpha$ is $p+q$, the sum of the multipliers of $n_1't$ and n_1t.

For, the general terms in u^ρ, u'^σ being

$$L\cos(ln_1't+\alpha), \text{ and } M\cos(mn_1t+\beta),$$

in which L and M are of the order l, m respectively, that of the product is $LM\cos(ln_1't \pm mn_1t+\alpha \pm \beta)$ which is of the order $l+m$.

(4) Again,

$$\cos k(\theta_1'-\theta_1) = \cos\{k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1)+k(v'-v)\}$$

$$=\cos k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1)\cos k(v'-v)$$

$$-\sin k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1)\sin k(v'-v).$$

Now, it is evident that the laws given above held also in the expansions of $\cos k(v'-v)$, and $\sin k(v'-v)$; therefore the general term of the expansion of $\cos k(\theta_1'-\theta_1)$ is to be obtained from the product of the sine or cosine of

$$k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1), \text{ by } P\,{}^{\cos}_{\sin}(Qt+\gamma),$$

where P is of the order given by these laws;

therefore, since $\quad pn_1't-qn_1t+\alpha$

$$=k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1)-\{(k\pm p)n_1't-(k\pm q)n_1t+\beta\},$$

$$\text{or, } k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1)+\{(p-k)n_1't-(q-k)n_1t+\gamma\}.$$

Hence the term in R whose argument is $pn_1't-qn_1t+\alpha$ is a series of terms whose orders are $k \pm p+k \pm q$ for all possible values of k from 0 to ∞.

The smallest value of $k+p+k+q$ is $p+q$, but since $k \sim p+k \sim q = 2k-(p+q)$ or $p \sim q$, its smallest value is $p \sim q$; therefore the order of such terms is $p \sim q$.

Order of the term $P\cos(pn_1't+qn_1t+\alpha)$.

33. In the expansion of powers and products of powers of u, v, $\tan\lambda$, u', v', $\tan\lambda'$, the order of a term whose argument is $(pn_1'\pm qn_1)\,t+\alpha$ is $p+q$, the sum of the multipliers of $n_1't$ and n_1t.

Hence, as in the last article, the general term in the expansion of $\cos k(\theta_2'-\theta_2)$ is obtained from the product of the sine or cosine of $k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1)$ by $P\,{}^{\cos}_{\sin}(Qt+\gamma)$, where P is of the order given by the laws investigated above; therefore, since $\qquad pn_1't+qn_1t+\alpha$

$$=k(n_1't+\varepsilon_1'-n_1t-\varepsilon_1)\mp\{(k\mp p)\,n_1't+(k\pm q)\,n_1t+\beta\}.$$

The term in R whose argument is $pn_1't+qn_1t+\alpha$ is a series of terms whose orders are $(k\mp p)+(k\pm q)$ for all possible values of k from 0 to ∞.

The smallest value of this expression is $p+q$; therefore the order of such terms is $p+q$.

34. By the aid of these properties of the function R, we are in a condition to select those terms of the higher orders of the small quantities, which, in the process of solution of the equations of motion, are of sufficient importance to be employed in the calculation of the disturbances of a planet by particular disturbing planets, and thus to avoid the accumulation of terms arising from the complete development of the disturbing function.

ON CERTAIN POINTS OF SINGULAR CURVATURE IN PLANE CURVES.

By E. WALKER, Trinity College, Cambridge.

It is my purpose in the present communication to draw attention to a difficulty which occurs in the consideration of a certain class of points of singular curvature; a difficulty which no writer, that I am acquainted with, has explained, certainly none of those whose works are current in this University.

The difficulty I allude to may be stated thus:

Let $y=f(x)$ be the equation to any plane curve, and suppose that when $x=a$, $\frac{d^2y}{dx^2}=\infty$, whilst $\frac{dy}{dx}$ remains finite;

then at the point $x=a$ the curvature is infinitely great, and the question is, what meaning shall we attach to such a result?

In certain cases, as we know, such a point is a cusp, but only when the lowest fractional index in the expansion of $f(a+h)$ is of a particular form. In other cases the curve to all appearance is continuous at the point in question, and these are the cases which I now propose to consider.

Let us take as an example the curve whose equation is

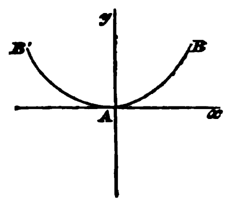
$$ay^3 = x^4 \text{ .... (1).}$$

At the origin we have

$$\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} = \infty,$$

and the curvature is infinite.

If we trace the curve in the usual way, we shall find its form to be that represented in the annexed figure.



What then do we mean by saying that the curvature at A is infinite?

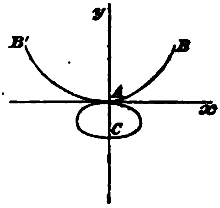
Of course it is easy to reply, that since we take the circle as the standard of curvature, we only mean, that the deflexion from the tangent is infinitely more rapid, in the immediate vicinity of the point A, for the curve whose equation is (1), than it can be for any circle; that is, that if we take *any circle whatever*, and place it so that the line Ax shall be a common tangent to the two curves, then, if their concavities are turned the same way, the circle must pass between the curve BAB' and the straight line Ax. But this does not get rid of the difficulty, which is, to shew how the above conclusion can hold for *all* circles (for instance, if the radius of the circle be indefinitely diminished), if the above be the correct form of the curve.

Perhaps the following considerations may throw some light on the difficulty.

Suppose that, instead of (1), we take the equation

$$ay^3 + aby^2 = x^4 \text{.................... (2),}$$

which becomes identical with (1) when $b = 0$. The form of the curve is given in the annexed figure, where $AC = b$.



Now draw a straight line touching the branch AB at some point between A and B, and make this tangent revolve till it touches the curve at some point between A and B'. It will be found that when the point of contact comes to A, it does not at once pass on to the branch AB', but comes back upon the loop AC, so that the tangent has to revolve through two right angles in passing from BA to AB'. In fact there are two cusps meeting at A. The above result will be true, however much we diminish AC, so that in the limit when (1) and (2) coincide, the tangent will turn abruptly through two right angles in passing from BA to AB'. We may now perhaps be better able to understand how it is that we cannot represent the curvature at A by reference to any circle whatever, and if it be objected that there is something arbitrary in the way in which we have added a term to the original equation, I reply that the more correct way of looking at the question is, that this equation, which is of the third degree in y, has been arbitrarily deprived of this term which properly belongs to it, and it constantly happens, that when we lower the generality of an expression, anomalies present themselves which can only be explained by recurring to the *more general* form, of which the *particular* form should be regarded as the limit.

The above example is only one out of many where the same difficulty occurs, but I regret that I cannot pursue the subject farther at present, though I hope to resume it at some future time. In the meanwhile my object will have been attained, if what I have said should lead to a closer examination of this class of singular points, which has not hitherto, as far as I am aware, attracted by any means the share of attention which it merits.

AN ATTEMPT TO DETERMINE THE TWENTY-SEVEN LINES UPON A SURFACE OF THE THIRD ORDER, AND TO DIVIDE SUCH SURFACES INTO SPECIES IN REFERENCE TO THE REALITY OF THE LINES UPON THE SURFACE.

By Dr. Schläfli, Professor of Mathematics at the University of Bern.
Translated by A. Cayley.

(Continued from p. 65).

I imagine to myself a homogeneous equation of the third order in the four point coordinates w, x, y, z, where all the twenty coefficients have any values whatever. From this may be calculated the function denoted above by R, which in the present case is a function of the degree 9. The surface $R=0$ will then meet the given basis surface of the third order $f=0$, in the twenty-seven lines of this surface. If therefore the equations $f=0$, $R=0$ are combined with any two linear equations

$$l=aw+bx+cy+dz=0,\quad l'=a'x+b'y+c'z+d'w=0,$$

it must be demonstrable that the resultant of the four functions f, R, l, l' can be (in respect to the indeterminate coefficients of the linear functions l, l') decomposed into twenty-seven factors of the form

$$\begin{vmatrix} \alpha a+\beta b+\gamma c, & d \\ \alpha a'+\beta b'+\gamma c', & d' \end{vmatrix} + \begin{vmatrix} \alpha', & \beta', & \gamma' \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}$$

where the constants α, β, γ, α', β', γ' satisfy the condition $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$. And then there will pass through the line corresponding to any such factor, the four planes

$$\gamma x-\beta y+\alpha' z=0,$$
$$-\gamma w+\alpha y+\beta' z=0,$$
$$\beta w-\alpha x+\gamma' z=0,$$
$$-\alpha' w-\beta' x-\gamma' y=0.$$

Suppose that one line of the given basis surface $f=0$ is known: and let the system of coordinates be transformed in such manner that two fundamental planes s, t, pass through the line in question. The equation of the surface will not contain any part not divisible by s or t, and it can therefore be reduced to the form $\begin{vmatrix} s, & S \\ t, & T \end{vmatrix} = 0$, where S and T denote polynomes of the second order. The basis surface contains therefore all the conics represented by the two equations $s + \lambda t = 0$, $S + \lambda T = 0$, where λ is an arbitrary constant. But λ can be so disposed of that the conic may break up into a pair of lines: the condition for this is, in regard to λ, of the fifth order; consequently, through each line of the basis there pass five planes, each of which intersects the basis in the three sides of a triangle, and the number of such planes is $\frac{27.5}{3} = 45$. Suppose that λ, μ are two different constants, satisfying the condition in question: the equation of the basis can then be brought under the form $\begin{vmatrix} s + \lambda t, & S + \lambda T \\ s + \mu t, & S + \mu T \end{vmatrix} = 0$, which may be denoted more simply by $\begin{vmatrix} u, & U \\ x, & X \end{vmatrix} = 0$. Here U, X are polynomes belonging to surfaces of the second order, which are respectively touched by the planes u, x. If p is the polynome of any other plane which touches both of the surfaces U, X, then there exists a constant α for which $U + \alpha pu$ breaks up into two factors, and in like manner a constant β for which $X + \beta pu$ breaks up into two factors. The plane p belongs to a developable of the fourth class, and has as such a single motion, i.e. its equation contains a single arbitrary parameter. We may therefore impose another condition, and write $\alpha = \beta$. Replace αp by the single letter p, and take D, Δ as symbols of the points in which the surfaces U, X are touched by the planes u, x respectively. Since then, each of the polynomes $U + pu$, $X + px$ breaks up into factors, it is clear that the equations

$$D(U + pu) = 0, \quad \Delta(X + px) = 0$$

will be satisfied identically. But obviously, $DU = au$, $\Delta X = bx$, where a and b are constants, and $Du = 0$, $\Delta x = 0$. The foregoing equations become therefore $a + Dp = 0$, $b + \Delta p = 0$, whence $\begin{vmatrix} a, & Dp \\ b, & \Delta p \end{vmatrix} = 0$, or if we please $\begin{vmatrix} Du, & uDp \\ \Delta x, & x\Delta p \end{vmatrix} = 0$, (the left-hand side divisible by ux) an equation which is homogeneous and linear in respect to the coefficients of p; that is, there exists a fixed point through which the simply moveable plane p always passes. The problem has therefore four solutions. And if we select at pleasure one of the four polynomes p which satisfy the required conditions, and write $U+pu=-yz$, $X+px=vw$, the equation of the basis becomes

$$\begin{vmatrix} u, & -yz \\ x, & vw \end{vmatrix} = uvw + xyz = 0.$$

The possibility of such a transformation might have been seen *à priori*, since the six linear polynomes u, &c., contain 18 ratios of coefficients, to which is to be added a constant factor contained in one of the products xyz, uvw; so that there are in all 19 disposable constants, which is precisely the number of conditions to be satisfied. We may call uvw a trihedral, and say that in the equation $uvw+xyz=0$, the basis is referred to a pair of trihedrals.

Six linear polynomes are connected together by two independent linear homogeneous equations. We may multiply one of these by an arbitrary factor, and add it to the second, and the relation so obtained will of course be satisfied. Let such a relation be

$$Au + Bv + Cw + Dx + Ey + Fz = 0,$$

where the coefficients are considered as containing a single arbitrary multiplier. It follows then, that

$$Au(Bv+Dx)(Cw+Dx) + Dx(Au+Ey)(Au+Fz)$$
$$= ABCuvw + DEFxyz,$$

consequently, that if $ABC=DEF$, the function on the left-hand side is a new expression for the polynome of the basis. The equation $ABC=DEF$ is, in regard to the arbitrary constant contained implicitly in the coefficients, of the degree 3, and gives therefore 3 solutions, which may be thus represented,

$$au + bv + cw + dx + ey + fz = 0, \quad abc = def,$$
$$a'u + b'v + c'w + d'x + e'y + f'z = 0, \quad a'b'c' = d'e'f',$$
$$a''u + b''v + c''w + d''x + e''y + f''z = 0, \quad a''b''c'' = d''e''f'',$$

there are thus in all 27 such transformations into trihedral pairs such as

$$au(bv+dx)(cw+dx) + dx(au+ey)(au+fz) = 0.$$

The original trihedral pair $uvw+xyz=0$, gives immediately nine lines. We may for shortness represent the line $(u=0, x=0)$ by $\overline{ux}$. We have besides, 18 systems such as $(au+dx=0,\ bv+ey=0,\ cw+fz=0)$, where the third equation is always a consequence of the two others; these systems represent the other 18 lines, which may be comprised in the following two schemes,

through	$\overline{ux}, \overline{vy}, \overline{wz}$	pass	$l,\ l',\ l''$
"	$\overline{uy}, \overline{vz}, \overline{wx}$	"	$m,\ m',\ m''$
"	$\overline{uz}, \overline{vx}, \overline{wy}$	"	$n,\ n',\ n''$

through	$\overline{ux}, \overline{vz}, \overline{wy}$	pass	$p,\ p',\ p''$
"	$\overline{uz}, \overline{vy}, \overline{wx}$	"	$q,\ q',\ q''$
"	$\overline{uy}, \overline{vx}, \overline{wz}$	"	$r,\ r',\ r''$

Two lines which belong to one and the same scheme do not intersect, when they are either lines represented by the same letter differently accented, or by different letters similarly accented; but they intersect when letters and accents are both different. And two lines belonging to different schemes, intersect when the accents are the same, and do not intersect when the accents are different.

Of the 45 triangle planes, 6 form the original trihedral pair, 27 more are represented by equations such as $au+dx=0$. We represent the plane $au+dx=0$, by (ux), the plane $a'u+d'x=0$, by $(ux)'$, and so in similar cases. The following scheme shews the three lines contained in each plane.

$\overline{ux},\ l,\ p$	$\overline{vx},\ n,\ r$	$\overline{wx},\ m,\ q$
$\overline{uy},\ m,\ r$	$\overline{vy},\ l,\ q$	$\overline{wy},\ n,\ p$
$\overline{uz},\ n,\ q$	$\overline{vz},\ m,\ p$	$\overline{wz},\ l,\ r$

and similarly with one or two accents. Finally the 12 remaining planes are 6 planes such as $lm'n''$, and 6 planes such as $pq'r''$, in the representation of which the accents may be omitted since the permutation of the letters is alone sufficient. The last mentioned planes admit of no very symmetrical representation. The plane (lmn) for example has among other forms of its equation the following,

$$\frac{cd'-c'd}{c'd}(au+dx)-\frac{bf'-b'f}{bf'}(bv+ey)=0.$$

Any two triangle planes which have no line in common, determine a third plane which forms with them a trihedral, and this again determines the other trihedral of the pair. There are thus in all $\frac{45.32}{6.2}=120$ trihedral pairs, that is, the problem to reduce the equation of the basis to the form $uvw+xyz=0$, is of the degree 120. Each trihedral pair gives immediately only nine lines. It is always possible to place together three trihedral pairs to give all the twenty-seven lines; and one pair determines by itself the other two pairs. There are thus in all 40 such triads of trihedral pairs, the following is a scheme of such triads,

1 triad

$$uvw+xyz,$$

$$(lmn)\,(mnl)\,(nlm)+(lnm)\,(nml)\,(mln),$$

$$(pqr)\,(qrp)\,(rpq)+(prq)\,(rqp)\,(qpr),$$

27 triads such as

$$u\,(vx)\,(wx)+x\,(uy)\,(uz),$$

$$(vy)'\,(wz)''\,(prq)+(vy)''\,(wz)'\,(pqr),$$

$$(vz)'\,(wy)''\,(lnm)+(vz)''\,(wy)'\,(lmn),$$

12 triads such as

$$u\,(lmn)\,(prq)+(ux)\,(uy)'\,(uz)'',$$

$$v\,(nlm)\,(rqp)+(vx)\,(vy)'\,(vz)'',$$

$$w\,(mnl)\,(qpr)+(wx)\,(wy)'\,(wz)''.$$

Choosing from each pair of any triad a single trihedral, we obtain nine planes which intersect the basis in all the 27 lines. The polynome of the ninth degree above represented by R, can therefore in 320 ways be combined with the polynome f of the basis, so as to break up into linear factors. An easier survey of the 27 lines of the basis f may be arrived at as follows. We have

$$2\,(uvw+xyz)=\begin{vmatrix} 0 & u & x \\ y & 0 & v \\ w & z & 0 \end{vmatrix}=0,$$

this equation by linear combinations of the lines and columns may be exhibited in the more general form

$$\begin{vmatrix} r, & s, & t \\ r', & s', & t' \\ r'', & s'', & t'' \end{vmatrix}=0,$$

where all the elements of the determinant are linear functions of u, v, w, x, y, z. Hence every point determined by a system of equations such as

$$p=\alpha r+\beta s+\gamma t=0,\quad p'=\alpha' r+\beta' s+\gamma' t=0,\quad p''=\alpha'' r+\beta'' s+\gamma'' t=0,$$

will lie on the basis, and conversely the ratios $\alpha : \beta : \gamma$ may be determined for a given point of the basis. But if the condition is imposed that the polynomes p, p', p'' shall be connected by an identical equation, such as $\kappa p + \kappa' p' + \kappa'' p'' = 0$, in other words, that the three planes shall intersect not in a point but in a line, we arrive at the condition that all the determinants of a rectangular matrix with three horizontal and three vertical lines, the elements of which are all linear homogeneous functions of α, β, γ, vanish. It is then clear that this problem has six solutions. If we assume for example that $\kappa p + \kappa' p' + \kappa'' p'' = 0$ is an identical equation, the equation of the basis may be exhibited in the form

$$\begin{vmatrix} 0, & \kappa s + \kappa' s' + \kappa'' s'', & \kappa t + \kappa' t' + \kappa'' t'' \\ p', & s' \quad , & t' \\ p'', & s'' \quad , & t'' \end{vmatrix} = 0,$$

which shows that each line $(p = 0,\ p' = 0,\ p'' = 0)$ corresponds to a line $(\Sigma\kappa r = 0,\ \Sigma\kappa s = 0,\ \Sigma\kappa t = 0)$ which it does *not* intersect. But if α, β, γ belong to a different solution, and the corresponding polynomes are denoted by q, q', q'', then we have

$$\begin{vmatrix} \Sigma\kappa q, & \Sigma\kappa s, & \Sigma\kappa t \\ q', & s', & t' \\ q'', & s'', & t'' \end{vmatrix} = 0$$

for the equation of the basis, and it is clear that now the two lines $(\Sigma\kappa q = 0,\ q' = 0)$ and $(\Sigma\kappa q = 0,\ \Sigma\kappa s = 0)$ intersect, since the systems have in common the equation $\Sigma\kappa q = 0$. Each of the six lines represented by a system such as $(p = 0,\ p' = 0,\ p'' = 0)$ intersects all the five non-corresponding lines given by a system such as $(\Sigma\kappa r = 0,\ \Sigma\kappa s = 0,\ \Sigma\kappa t = 0)$, and only the two corresponding lines do not intersect. I call such group of 12 lines of the basis a "double-six." It is clear that no two lines belonging to the same six intersect. The number of all the double-sixes is 36. For since each line is intersected by 10 other lines, there remain 16 lines by which it is not intersected. There are therefore $\dfrac{27 \cdot 16}{2} = 216$ pairs of lines which do not intersect. Through one of the lines of such a pair pass five lines which do not intersect the other line of the pair; this other line and the five lines form together a six, and these completely determine the other six of the double-six. But of such pairs of corresponding lines as the first-mentioned pair there are in the double-six only 6; consequently $\frac{216}{6}=36$ is the number of the double-sixes.

If now we start from the equation

$$\begin{vmatrix} 0, & u, & x \\ y, & 0, & v \\ w, & z, & 0 \end{vmatrix} = 0,$$

we have at once three solutions of the problem, to make the polynomes $\beta u + \gamma x$, $\alpha y + \gamma v$, $\alpha w + \beta z$ dependent on each other, namely $(\beta = 0,\ \gamma = 0)$, $(\alpha = 0,\ \gamma = 0)$, $(\alpha = 0,\ \beta = 0)$; the other three are obtained as follows: Suppose that

$$\kappa(\beta u + \gamma x) + \kappa'(\alpha y + \gamma v) + \alpha''(\alpha w + \beta z) = 0$$

is the identical relation between the three polynomes, and

$$Au + Bv + Cw + Dx + Ey + Fz = 0$$

the general identical relation, where A, &c. are to be considered as linear functions of a single disposable quantity. We must therefore write

$$A = \kappa\beta, \quad B = \kappa'\gamma, \quad C = \kappa''\alpha, \quad D = \kappa\gamma, \quad E = \kappa'\alpha, \quad F = \kappa''\beta,$$

which give $ABC = DEF$. This equation admits, as we know already, of three solutions. And preserving the former notations, we thus arrive at the double-six

$$\begin{pmatrix} \overline{uz}, & \overline{vx}, & \overline{wy}, & l, & l', & l'' \\ \overline{vy}, & \overline{wz}, & \overline{ux}, & n, & n', & n'' \end{pmatrix},$$

where no two lines of the same horizontal row and no two lines of the same vertical row intersect, but any two lines otherwise selected do intersect.

By means of the double-sixes we arrive, as already noticed, at an easy survey of the 27 lines and 45 planes of the basis. For represent a double-six by

$$\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ b_1, & b_2, & b_3, & b_4, & b_5, & b_6 \end{pmatrix},$$

the two intersecting lines a_1, b_2 belong to a triangle which I represent by 12 and its third side by c_{12}. This third side c_{12} forms with a_2, b_1 a triangle which I represent by 21. We have thus fifteen lines c, each of which intersects only those four lines a, b, the suffixes of which belong to the pair of numbers forming the suffix of the c. And any two c's, the suffixes of which have a number in common, do not intersect; but two c's, the suffixes of which have no number in common, do intersect. There are consequently triangles such as c_{12}, c_{34}, c_{56} which may be represented by (12, 34, 56), where as well the numbers *inter se* of each pair, as the three pairs *inter se*, may be permuted. We have therefore 30 triangles such as 12, and 15 triangles such as (12, 34, 56), in all 45. Finally there are 10 trihedral pairs such as

$$(12)\,(23)\,(31) + (13)\,(32)\,(21)$$
$$(45)\,(56)\,(64) + (46)\,(65)\,(54)$$
$$(14,25,36)(15,26,34)(16,24,35) + (14,26,35)(16,25,34)(15,24,36)$$

and 30 trihedral pairs such as

$$(35)\,(46)\,(12,\,36,\,45) + (36)\,(45)\,(12,\,35,\,46)$$
$$(51)\,(62)\,(16,\,25,\,34) + (52)\,(61)\,(15,\,26,\,34)$$
$$(13)\,(24)\,(14,\,23,\,56) + (14)\,(23)\,(13,\,24,\,56)$$

The double-sixes give rise to the remark that there is here exposed to view an apparently very elementary theorem which may be thus enuntiated: "Drawing at pleasure five lines a, b, c, d, e which meet a line F, then may any four of the five lines be intersected by another line besides F. Suppose that A, B, C, D, E are the other lines intersecting (b, c, d, e), (c, d, e, a), (d, e, a, b), (e, a, b, c), and (a, b, c, d) respectively. Then A, B, C, D are intersected by the line e; there must be another line f intersecting these four lines, and this line will of itself intersect the remaining line E; i.e. there will be a line f intersecting the five lines A, B, C, D, E." Is there, for this elementary theorem, a demonstration more simple than the one derived from the theory of cubic forms?

When the equation of the cubic surface referred to a real system of coordinate axes, has all its coefficients real, it is easy to see that the surface will be real. The question however arises, how many of the 27 lines and 45 planes may be imaginary? The complete investigation would be tedious, and I content myself in giving a mere survey of the species into which the general surface of the third order divides itself in regard to the reality of the 27 lines. There are only the five species following:

A. All the 27 lines and 45 planes are real.

B. 15 lines and 15 planes are real. The twelve imaginary lines form a double-six, where each line of the one six is conjugate to the corresponding and therefore not intersecting line of the other six, wherefore none of the imaginary lines have a real point. Any two pairs of corresponding imaginary lines are intersected by a real line; and as many ways as the double-six can be divided into thrice two such pairs, in so many ways do the corresponding real lines form a triangle, viz. there are fifteen real triangles.

C. 7 lines and 5 planes are real: namely, through a real line there pass 5 real planes, but of these three only contain real triangles, in each of the other two the triangle consists of the original real line and two imaginary lines meeting in a real point.

D. 3 lines and 13 planes are real: namely, there is one real triangle, and through each side there pass (besides the plane of the triangle) 4 real planes.

E. 3 lines and 7 planes are real: namely, there is a real triangle, and through each side there pass (besides the plane of the triangle) 2 real planes.

With respect to the reality or non-reality of the six linear polynomes in the expression $uvw + xyz$, which is equivalent to a given cubic polynome with real coefficients, the following 13 cases have to be distinguished. I call them forms of the trihedral pair $uvw + xyz = 0$, and I shew in the following enumeration to which species of cubic surface each form belongs: instead of linear polynome the word plane may be used.

1°. All the six planes of the trihedral pair are real. This form occurs only in the species *A* and *B*.

2°. u and x, v and y, w and z are conjugate to each other; that is, the two trihedrals of the pair are imaginary and conjugate. In *B* and *C*.

3°. u, v, w, x are real, y and z conjugate. In *D* and *E*.

4°. u and x are real, v and w, and y and z conjugate to each other. In *C* and *E*.

5°. u and x are real, the four others imaginary, but no two of them conjugate: but v and w have their real line in x, and y and z their real line in u. (Every imaginary plane contains of course a real line). In *B* and *C*.

6°. u and x are real, the four others imaginary and no two of them conjugate: and u alone intersects y, z in real lines. In *C* and *E*.

7°. u and x are real, the four others imaginary and not conjugate. Neither u nor x have a real triangle. In *D* and *E*.

8°. u and x are conjugate, the four others are imaginary and not conjugate; v and y have a real point in common, and so have w and z. In *C* and *E*.

9°. u is real, the five other planes are imaginary and not conjugate, u intersects x in a real line, and y, z in conjugate lines. And y alone has with each of the planes v and w, a real point in common. In E.

10°. All the six planes are imaginary and not conjugate; u and x have in common a real point, v and y a real line, and w and z a real line. In C.

11°. All the six planes are imaginary and not conjugate, each plane of the one trihedral has in common with each plane of the other trihedral a real point. In D.

12°. All the six planes are imaginary and not conjugate; u has in common with x a real point, and also with y, and also with z; moreover x has a real point in common with v, and also with w.

13°. All the six planes are imaginary and not conjugate; u has a real point in common with x, and so also v with y, and w with z. In E.

If in any one of these thirteen forms the particular complete character of each of the six linear polynomes is represented explicitly, and then the transformation is undertaken of this form into another trihedral pair, it often happens that a root of the cubic equation which has to be solved for this purpose can be rationally represented by the constants of the form without the necessity of extracting a cube root. Two trihedral pair forms thus easily transformable the one into the other may be termed *equivalent;* when the one of them presents itself in any two species of the surface, the other also presents itself in the same two species. It is only the two other roots of the above mentioned cubic equation $ABC = DEF$ which decide, according as they happen to be real or imaginary, to which of the two species the surface belongs, and they give rise to a transformation complicated with a square root; trihedral pairs thus transformed into each other, on account of the possible transition from one species into a different one, I call *non-equivalent;* the more so that the discussion of the one form does not render unnecessary that of the other. In this sense

The forms	2°, 5°	are equivalent	and occur in	B and C,
"	4°, 6°, 8°	"	"	C and E,
"	3°, 7°	"	"	D and E,
"	9°, 12°, 13°	"	"	E,

while, on the contrary, the following forms are each of them isolated, viz. 1° in A and B, 10° in C, and 11° in D.

The forms of the triads of trihedral pairs arrange themselves as follows:

A has 40 triads (1, 1, 1).

B has 10 triads (1, 2, 2) and 30 triads (5, 5, 5).

C has 4 triads (2, 2, 4), 12 triads (5, 8, 8), and 24 triads (6, 10, 10).

D has 16 triads (3, 11, 11) and 24 triads (7, 7, 7).

E has 2 triads (4, 4, 4), 4 triads (3, 13, 13), 6 triads (7, 8, 8), 12 triads (6, 12, 12), and 16 triads (9, 9, 9).

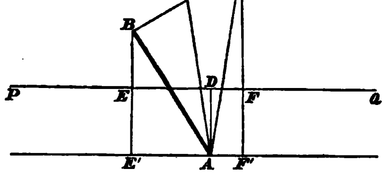
In conclusion I remark that the double-sixes play a part in the theory of the nodes of a cubic surface. I call "node" any point (w, x, y, z) of an algebraical surface $f(w, x, y, z)=0$, for which $Df=0$ is satisfied for all values of the four elements of the differentiation system D, and "proper node" a point at which the cone of the second order represented by $D^2f=0$ does not break up into a pair of planes. If a surface of the 3rd order $f=0$ has a proper node (w, x, y, z), then the six lines passing through such node and represented by the equations $D^2f=0$, $D^3f=0$ form a double-six, in which each two corresponding (non-intersecting) lines of the two sixes coincide.

NOTE ON THE RELATION BETWEEN THE DISTANCES OF A STRAIGHT LINE FROM THREE GIVEN POINTS.

By N. M. Ferrers, Gonville and Caius College, Cambridge.

In Salmon's *Higher Plane Curves*, Art. IX., p. 10, an investigation is given of the relation existing between the tangential coordinates of a straight line, that is, between its distances from three arbitrarily chosen points.

The result is there arrived at through the intervention of Cartesian coordinates, but it appears desirable to place the system of tangential coordinates on an independent basis, and therefore to supply a proof of the relation in question not depending on any other system.



Let A, B, C be the three points of reference, PQ any straight line, draw AD, BE, CF, severally perpendicular to PQ. Then, in accordance with the convention with respect to the use of the negative sign, if AD be considered positive, BE and CF, lying wholly *without* the angles ABC, ACB respectively, will be negative. Let then $AD = \alpha$, $BE = -\beta$, $CF = -\gamma$. Through A draw $E'F'$ parallel to PQ, produce BE, CF to meet $E'F'$ in E', F' respectively, then

$$BE' = -(\beta + \alpha), \quad CF' = -(\gamma + \alpha).$$

Bisect the angle BAC by the straight line AO, and let $OAD = \theta$.

Let $$BC = a, \quad CA = b, \quad AB = c.$$

Then $$-\frac{\beta + \alpha}{c} = \frac{BE'}{BA} = \sin BAE' = \cos BAD = \cos\left(\frac{A}{2} + \theta\right).$$

Similarly $$-\frac{\gamma + \alpha}{b} = \cos\left(\frac{A}{2} - \theta\right),$$

therefore $$-\frac{1}{2\cos\frac{A}{2}}\left(\frac{\beta + \alpha}{c} + \frac{\gamma + \alpha}{b}\right) = \cos\theta,$$

$$\frac{1}{2\sin\frac{A}{2}}\left(\frac{\beta + \alpha}{c} - \frac{\gamma + \alpha}{b}\right) = \sin\theta;$$

therefore, adding squares and simplifying,

$$\frac{1}{\sin^2 A}\left\{\frac{(\beta + \alpha)^2}{c^2} + \frac{(\gamma + \alpha)^2}{b^2}\right\} - \frac{2\cos A}{\sin^2 A}\,\frac{(\beta + \alpha)(\gamma + \alpha)}{bc} = 1;$$

therefore

$$b^2(\beta + \alpha)^2 + c^2(\gamma + \alpha)^2 - 2bc\cos A(\beta + \alpha)(\gamma + \alpha) = b^2c^2\sin^2 A,$$

or $$b^2(\beta + \alpha)^2 + c^2(\gamma + \alpha)^2 - (b^2 + c^2 - a^2)(\beta + \alpha)(\gamma + \alpha) = 4K^2,$$

K denoting the area of the triangle ABC; a result which may be put into the symmetrical form,

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - (b^2 + c^2 - a^2)\beta\gamma - (c^2 + a^2 - b^2)\gamma\alpha - (a^2 + b^2 - c^2)\alpha\beta = 4K^2,$$

which is substantially identical with that given by Mr. Salmon, in the article above referred to.

FACTORIAL NOTATION.

By H. W. Elphinstone, Trinity College, Cambridge.

The object of the following paper is to call the attention of Mathematicians to the advantages of the Factorial Notation. This notation has, according to Professor De Morgan, met with little attention in England. The only English works in which I have found it made use of are Professor De Morgan's *Differential Calculus*; Nicholson's *Essays*; and a paper in the *Cambridge Transactions*, by Henry Warburton. It seems surprising that this notation should have met with such universal disregard, for it is useful not in Elementary Algebra only, but it plays a part in the Calculus of Finite Differences, similar to that which the notation of Indices does in the Differential Calculus.

1. *Definition*. The product of any number of factors in Arithmetic progression is called a factorial, and is expressed by the following notation:

$$x(x+a)\ldots\{x+(n-1)a\}=x^{n|a}$$

in which x the first factor is called the *base*, n the number of factors is called the *factorial index*, a the common difference between the factors is called the *factorial difference*.

It follows from our notation that

$$x(x-a)\ldots\{x-(n-1)a\}=x^{n|-a}.$$

Fractions of the form $\dfrac{x^{n|-1}}{1^{n|1}}$, $\dfrac{x^{n|1}}{1^{n|1}}$ occur frequently in analysis, and may be called *divided* factorials.

2. Reduction of factorials with a difference $\pm a$ to a difference ± 1.

$$\begin{aligned}x^{n|\pm a}&=x(x\pm a)(x\pm 2a)\ldots\{x\pm(n-1)a\}\\&=a^n\frac{x}{a}\left(\frac{x}{a}\pm 1\right)\left(\frac{x}{a}\pm 2\right)\ldots\left\{\frac{x}{a}\pm(n-1)\right\}\\&=a^n\left(\frac{x}{a}\right)^{n|\pm 1}.\end{aligned}$$

Since all factorials may in this manner be transformed so as to have a difference ± 1, I shall generally confine my attention to factorials of the form $x^{n|\pm 1}$.

3. In the factorial $x^{n|\pm 1}$ suppose that the common difference has a different sign from x the base, and that n is numerically greater than x, there will be a factor equal to zero, so that the whole factorial will be equal to zero.

There are various other properties which will suggest themselves very readily; or they may be found in "A Treatise on Factorial Analysis, &c.," by Thomas Tate. But as I only desire to explain the factorial notation, I shall not dwell upon them.

4. *Basic Transformation.* Divided factorials with a positive difference may be made to undergo a transformation which is very often of the greatest use.

$$\frac{x^{n|1}}{1^{n|1}} = \frac{1^{x-1|1}.x^{n|1}}{1^{x-1|1}\,1^{n|1}} = \frac{1^{x-1+n|1}}{1^{x-1|1}.1^{n|1}}$$

$$= \frac{(n+1)^{x-1|1}}{1^{x-1|1}}.$$

Hence the rule. Take the index increased by unity for a new base, and the base decreased by unity for a new index.

The difficulty that occurs in remembering this rule is to remember which of the two, the index or base, is diminished. This can be done by noticing that the new denominator of the fraction consists of the missing factors of the old numerator.

When the common difference is negative the formula is not so simple, for we have

$$\frac{x^{n|-1}}{1^{n|1}} = \frac{1^{x|1}}{1^{n|1}.1^{x-n|1}} = \frac{(n+1)^{x-n|1}}{1^{x-n|1}}.$$

As an example of this transformation we may reduce the coefficients in $(1+x)^{-n}$ to a common denominator.

For

$$(1+x)^{-n} = 1 - nx + \frac{n^{2|1}}{1^{2|1}}x^2 - \ldots + (-1)^r\frac{n^{r|1}}{1^{r|1}}x^r + \&c.$$

$$= \frac{n^{0|1}}{1^{0|1}} - \frac{n^{1|1}}{1^{1|1}}x + \frac{n^{2|1}}{1^{2|1}}x^2 - \ldots + (-1)^r\frac{n^{r|1}}{1^{r|1}}x^r + \&c.$$

$$= \frac{1^{n-1|1}}{1^{n-1|1}} - \frac{2^{n-1|1}}{1^{n-1|1}}x + \frac{3^{n-1|1}}{1^{n-1|1}}x^2 - \ldots + (-1)^r\frac{(r+1)^{n-1|1}}{1^{n-1|1}}x^r + \&c.$$

5. To interpret a factorial with a negative index

$$x^{m-n|1} = \frac{1^{x-1+m-n|1}}{1^{x-1|1}}$$

$$= \frac{1^{x-1+m|1}}{1^{x-1|1}.(x-1+m)^{n|-1}},$$

or putting $\qquad m = 0,$

$$x^{-n|1} = \frac{1^{x-1|1}}{1^{x-1|1}} \frac{1}{(x-1)^{n|-1}}$$

$$= \frac{1}{(x-1)^{n|-1}},$$

$$x^{m-n|-1} = \frac{1^{x|1}}{1^{x-m+n|1}};$$

therefore $\qquad x^{-n|-1} = \frac{1^{x|1}}{1^{x+n|1}} = \frac{1}{(x+1)^{n|1}}.$

Hence the rule. A factorial, with a negative index, is equal to the reciprocal of a factorial, whose base is the original base increased or diminished by unity according as the common difference is negative or positive, and whose index and common difference are those of the proposed factorial with changed signs.

Observe the analogy to the expression $x^{-n} = \frac{1}{x^n}$.

These two expressions may also be put under the forms

$$x^{-n|1} = \frac{1}{(x-n)^{n|1}},$$

$$x^{n|-1} = \frac{1}{(x+n)^{n|-1}}.$$

If, instead of making m equal to zero, we put $m = n$, we shall have

$$x^{0|1} = \frac{1^{x-1+n-n|1}}{1^{x-1|1}} = 1,$$

which may be compared with the expression $x^0 = 1$.

6. The analogy between factorials and powers is no where seen more distinctly than when we proceed to take their differences.

$$\Delta x^{n|1} = (x+1)^{n|1} - x^{n|1} = (x+1)^{n-1|1}(x+n-x)$$

$$= n(x+1)^{n-1|1},$$

$$\Delta x^{n|-1} = (x+1)^{n|-1} - x^{n|-1} = x^{n-1|-1}\{x+1-(x-n+1)\}$$

$$= nx^{n-1|-1}.$$

Hence the rule. Multiply by the index, subtract unity from the index to form a new index, and if necessary add unity to the base, so that the highest factor may remain what it was before.

$$\Delta\frac{1}{x^{n|1}}=\frac{1}{(x+1)^{n|1}}-\frac{1}{x^{n|1}}=\frac{1}{(x+1)^{n-1|1}}\left(\frac{1}{x+n}-\frac{1}{x}\right)=\frac{-n}{x^{n+1|1}},$$

$$\Delta\frac{1}{x^{n|-1}}=\frac{1}{(x+1)^{n|-1}}-\frac{1}{x^{n|-1}}$$

$$=\frac{1}{x^{n-1|-1}}\left(\frac{1}{x+1}-\frac{1}{x-n+1}\right)=\frac{-n}{(x+1)^{n+1|-1}}.$$

Hence the rule, multiply by the index with changed sign, add unity to the index: and if necessary, add unity to the base, so that the lowest factor may remain what it was before. Compare these formulas with

$$\frac{d}{dx}x^n=nx^{n-1},\quad \frac{d}{dx}\frac{1}{x^n}=-\frac{n}{x^{n+1}}.$$

The difficulty in remembering these rules, is to remember in what case the base remains the same. This can be done by noticing that the most important factor, i.e. the greatest in $x^{n|\pm1}$ and the least in $\frac{1}{x^{n|\pm1}}$ is retained in each case.

We shall have

$$\Delta\,(x-1)^{n+1|1}=(n+1)\,x^{n|1};$$

therefore $\quad\Sigma\, x^{n|1}=\frac{1}{n+1}(x-1)^{n+1|1},$

$$\Delta\, x^{n+1|-1}=(n+1)\,x^{n|-1},$$

therefore $\quad\Sigma\, x^{n|-1}=\frac{1}{n+1}\,x^{n+1|-1},$

$$\Delta\frac{1}{x^{n-1|1}}=\frac{-(n-1)}{x^{n|1}},$$

therefore $\quad\Sigma\frac{1}{x^{n|1}}=\frac{-1}{n-1}\,\frac{1}{x^{n-1|1}},$

$$\Delta\frac{1}{(x-1)^{n-1|-1}}=\frac{-(n-1)}{x^{n|-1}},$$

therefore $\quad\Sigma\frac{1}{x^{n|-1}}=\frac{-1}{n-1}\,\frac{1}{(x-1)^{n-1|-1}}.$

The rules are. To find the finite integral of $x^{n|\pm1}$, increase the index by unity for a new index, and divide by the index so increased, diminishing the base by unity if necessary, so that the greatest factor may remain the same.

To find the integral of $\dfrac{1}{x^{n|\pm1}}$ subtract unity from the index for a new index, divide by the new index with its sign changed, diminishing the base if necessary, so that the least factor may remain the same.

As in the former case, we may observe that the most important factor is retained unaltered.

Compare this with

$$\int x^n dx = \frac{1}{n+1}\, x^{n+1}, \quad \int \frac{dx}{x^n} = \frac{-1}{(n-1)\, x^{n-1}}.$$

7. Let Δ' denote the difference taken with respect to n.

Then
$$\Delta' \frac{x^{n|1}}{1^{n|1}} = \Delta' \frac{(n+1)^{x-1|1}}{1^{x-1|1}}$$
$$= \frac{(n+2)^{x-2|1}}{1^{x-2|1}}$$
$$= \frac{(x-1)^{n+1|1}}{1^{n+1|1}}$$
$$= \Sigma \frac{x^{n|1}}{1^{n|1}}$$

a curious formula shewing that the operations Δ' and Σ are equivalent.

8. Suppose that

$$E = 1 + \Delta.$$

Then
$$E^m f(x) = (1+\Delta)^m f(x),$$
$$= \left(1 + m\Delta + \frac{m^{2|-1}}{1^{2|1}}\, \Delta^2 + \&c.\right) f(x),$$

$$f(x+m) = f(x) + m\Delta f(x) + \frac{m^{2|-1}}{1^{2|1}}\, \Delta^2 f(x) + \&c.,$$

similarly, $f(x-m) = f(x) - m\Delta f(x) + \dfrac{m^{2|1}}{1^{2|1}}\, \Delta^2 f(x) - \&c.,$

or putting $x=0$,

$$f(m)=f(0)+m\Delta f(0)+\frac{m^2|^{-1}}{1^2|^1}\Delta^2 f(0)+\&c.,$$

$$f(-m)=f(0)-m\Delta f(0)+\frac{m^2|^1}{1^2|^1}\Delta^2 f(0)-\&c.,$$

known formulas analogous to those of Taylor and Maclaurin.

9. We can also obtain other expressions for $f(x)$ in terms of $m^r|^{\pm 1}$.

For since $\Delta = E-1$,

and $\Delta = \dfrac{1}{\Sigma}$.

It follows that

$$1=\Sigma^m (E-1)^m,$$
$$1=\Delta^m (E-1)^{-m},$$
$$1=E^m (1+\Delta)^{-m},$$
$$1=E^{-m} (1+\Delta)^m;$$

from these four expressions we get

$$f(x)=\Sigma^m f(x+m)-m\Sigma^m f(x+m-1)+\frac{m^2|^{-1}}{1^2|^1}\Sigma^m f(x+m-2)-\&c.,$$

$$f(x)=\Delta^m f(x-m)+m\Delta^m f(x-m+1)+\frac{m^2|^1}{1^2|^1}\Delta^m f(x+m-2)+\&c.,$$

$$f(x)=f(x+m)-m\Delta f(x+m)+\frac{m^2|^1}{1^2|^1}\Delta^2 f(x+m)-\&c.,$$

$$f(x)=f(x-m)+m\Delta f(x-m)+\frac{m^2|^{-1}}{1^2|^1}\Delta^2 f(x-m)+\&c.$$

Expressions for $f(x)$ in terms of the factorials of some other quantity m.

10. The formulas of Art. 8 afford an easy proof of the binomial theorem of factorials.

Expanding the left-hand side of the following equations by those formulas we have

$$(x+m)^n|^1=x^n|^1+n(x+1)^{n-1}|^1m+\frac{n^2|^{-1}}{1^2|^1}(x+2)^{n-2}|^1m^2|^{-1}+$$
$$+\frac{n^r|^{-1}}{1^r|^1}(x+r)^{n-r}|^1m^r|^{-1}+\&c.,$$

$$(x+m)^n|^{-1} = x^n|^{-1} + \quad nx^{n-1}|^{-1}m + \frac{n^2|^{-1}}{1^2|^1} x^{n-2}|^{-1}m^2|^{-1} + \dots$$

$$+ \frac{n^r|^{-1}}{1^r|^1} x^{n-r}|^{-1}m^r|^{-1} + \&c.,$$

$$(x-m)^n|^1 = x^n|^1 - n(x+1)^{n-1}|^1 m + \frac{n^2|^{-1}}{1^2|^1}(x+2)^{n-2}|^1 m^2|^1 - \&c.,$$

$$+ (-1)^r \frac{n^r|^{-1}}{1^r|^1}(x+r)^{n-r}|^1 m^r|^1 + \&c.,$$

$$(x-m)^n|^{-1} = x^n|^{-1} - \quad nx^{n-1}|^1 m + \frac{n^2|^{-1}}{1^2|^1} x^{n-2}|^1 m^2|^1 - \&c.$$

$$+ (-1)^r \frac{n^r|^{-1}}{1^r|^1} x^{n-r}|^{-1} m^r|^1 + \&c.$$

11. Let ∇ denote $1 - E^{-1}$, a difference used by German writers. Then ∇^{-1} is the function generally denoted by S. And we have

$$\begin{aligned}\nabla x^n|^1 &= x^n|^1 - (x-1)^n|^1 \\ &= x^{n-1}|^1 \{x+n-1-(x-1)\} \\ &= nx^{n-1}|^1, \\ Sx^n|^1 &= \frac{1}{n+1} x^{n+1}|^1 \\ \nabla x^n|^{-1} &= x^n|^{-1} - (x-1)^n|^{-1} \\ &= (x-1)^{n-1}|^{-1}\{x-(x-n)\} \\ &= n(x-1)^{n-1}|^{-1} \\ Sx^n|^{-1} &= \frac{1}{n+1}(x+1)^{n+1}|^{-1}.\end{aligned}$$

Formulas that are occasionally of use.

I have now, I hope, written enough to call attention to the Factorial Notation. I hope at some future time to make a communication to the *Journal*, containing some other properties of Factorials.

Trinity College,
May, 1857.

H. W. E.

ON A PROPOSITION IN ATTRACTIONS.

By E. J. Routh, M.A., St. Peter's College, Cambridge.

Let a number of particles $m_1 m_2 \ldots$ attract another particle P with forces which vary inversely as the $n+1^{th}$ power of the distance, and let

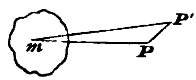
$$V = \Sigma \frac{m}{r^n},$$

then, if dV be the difference between the value of V at any point P, and that at an adjacent point P', where $PP' = dx$, the attraction on P resolved in the direction PP' is the limit of

$$X = \frac{1}{n} \frac{dV}{dx},$$

excepting when $n = 0$.

For let m be one of the attracting particles and let



$mP = r$, and $\theta =$ the angle mP makes with PP', then ultimately, $mP' = r + dx \cos\theta$. And the increase of V, so far as it depends on m, will be

$$\frac{m}{(r + dx \cos\theta)^n} - \frac{m}{r^n} = -\frac{m}{r^{n+1}} \cos\theta . ndx;$$

therefore $$\frac{1}{n} \frac{dV}{dx} = -\Sigma . \frac{m \cos\theta}{r^{n+1}}$$

= resolved part of the force in the direction PP'.

This quantity V is called the potential at P.

Def. Let two surfaces A, A' be so connected with an external point S, that if any radius vector SPP' be drawn, cutting them both the product $SP . SP'$ is a constant quantity κ^2, then we may call these surfaces *reciprocal* surfaces, and the points P, P' *reciprocal* points.

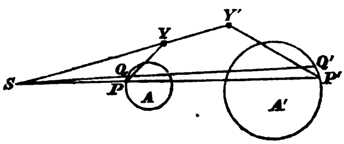
Prop. *To compare the attractions of two reciprocal bodies on two reciprocal points.*

Take any two reciprocal points Y, Y' not on the surfaces, then $SY . SY' = \kappa^2$. Let us compare the values of the potentials V, V' of the two surfaces A, A' at the two points Y, Y'.

With vertex S describe a cone cutting off from the surfaces two elementary areas PQ, $P'Q'$. Let the law of density over the two surfaces be represented respectively by

$$\overline{\frac{\lambda}{SP}}\Big|^{m} \text{ and } \overline{\frac{\lambda}{SP'}}\Big|^{m'},$$

and let the law of attraction be $\frac{\text{mass}}{(\text{dist})^{n+1}}$.



Then $$\frac{\text{potential of } PQ \text{ at } Y}{\text{potential of } P'Q' \text{ at } Y'} = \frac{\frac{\text{area } PQ}{PY^n} \cdot \frac{\lambda^m}{SP^m}}{\frac{\text{area } P'Q'}{P'Y'^n} \cdot \frac{\lambda^{m'}}{SP'^{m'}}}.$$

But because $SP.SP' = \kappa^2 = SQ.SQ'$ the triangles PSQ, $Q'SP'$ are similar, and therefore

$$\frac{\text{area } PQ}{\text{area } P'Q'} = \frac{SP^2}{SP'^2}.$$

Again, P and Y being any two points, and P', Y' the reciprocal points, the triangles SPY, $SY'P'$ are similar; and

therefore $$\frac{PY}{P'Y'} = \frac{SY}{SP'} = \frac{\kappa^2}{SP'.SY'}.$$

Hence substituting, the ratio of the two potentials becomes

$$= \frac{SP^2}{SP'^2} \cdot \frac{SP'^n}{SY^n} \cdot \frac{SP'^{m'}}{SP^m} \cdot \lambda^{m-m'}$$

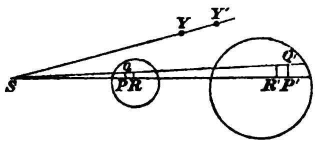
$$= \frac{\overline{SP'}\big|^{m+m'+n-4}}{\kappa^{2m-4}.SY^n} \cdot \lambda^{m-m'}.$$

Hence, provided $m + m' + n = 4$, this ratio is constant, and the same for all the elements. Hence, for the whole surfaces,

$$\frac{V \text{ of } A \text{ at } Y}{V \text{ of } A' \text{ at } Y'} = \frac{\lambda^{m-m'}}{\kappa^{2m-4}} \cdot \frac{1}{SY^n}.$$

Thus, when the potential of one surface is known, that of the other may be found.

If instead of comparing the potentials of the *surfaces* we wish to compare those of the *volumes*, the mode of proceeding will be the same as before.



Take two reciprocal elements PQR, $P'Q'R'$ as before,

$$\frac{\text{potential of } PQ \text{ at } Y}{\text{potential of } P'Q' \text{ at } Y'} = \frac{\frac{\text{vol } QR}{PY^n} \cdot \frac{\lambda^m}{SP^m}}{\frac{\text{vol } Q'R'}{P'Y'^n} \cdot \frac{\lambda^{m'}}{SP'^{m'}}},$$

and because $SP.SP' = \text{constant}$, differentiating

$$SP.d(SP') + SP'.d(SP) = 0,$$

or

$$SP.P'R' = SP'.PR,$$

and, as before, the areas PQ, $P'Q'$ are in the ratio $SP^2 : SP'^2$, hence

$$\frac{\text{vol } PQ}{\text{vol } P'Q'} = \frac{SP^3}{SP'^3},$$

and the ratio of the potentials will now be found to be

$$= \frac{SP'^{m+m'+n-6}}{\kappa^{2m-6}} \cdot \frac{\lambda^{m-m'}}{SY^n},$$

hence, provided $m + m' + n = 6$, this ratio is constant, and the same for all elements, and therefore for the whole volumes;

therefore $$\frac{V \text{ of } A \text{ at } Y}{V' \text{ of } A' \text{ at } Y'} = \frac{\lambda^{m-m'}}{\kappa^{2m-6}} \cdot \frac{1}{SY^n}.$$

If instead of surfaces or volumes we wished to compare the potentials of two *arcs* A, A', then we may prove, in the same way, that provided $m + m' + n = 2$,

$$\frac{V \text{ of } A \text{ at } Y}{V' \text{ of } A' \text{ at } Y'} = \frac{\lambda^{m-m'}}{\kappa^{2m-2}} \cdot \frac{1}{SY^n}.$$

From these results we may infer the two following properties.

1. If the potential of an arc, surface, or volume A' be constant $(= V_0')$ the density being supposed to follow the law

$\frac{\lambda^{m'}}{SP'^{m'}}$, then the potential of the reciprocal arc, surface, or volume will be the same as that of a certain mass M collected at S, provided the density follows the law $\frac{\lambda^{m}}{SP^{m}}$ where $m + m' + n$ is either 2, 4, or 6, according as the bodies considered are arcs, surfaces, or volumes.

2. If the potential of an arc, surface, or volume A' be the same as that of a certain mass M' collected at a point O', then the potential of the reciprocal will be the same as that of a certain mass M collected at O, where O, O' are reciprocal points. And this mass M may be found thus. By the previous formulæ,

$$V \text{ or } A \text{ on } Y = \frac{\lambda^{m-m'}}{\kappa^{2m-\mu}} \cdot \frac{M'}{O'Y'^{n}} \cdot \frac{1}{SY^{n}}$$

$$= \frac{\lambda^{m-m'}}{\kappa^{2m-\mu}} \cdot \frac{M'}{OY^{n}} \cdot \frac{SY^{n}}{O'S^{n}} \cdot \frac{1}{SY^{n}};$$

therefore

$$M = \frac{\lambda^{m-m'} . M' . SO^{n}}{\kappa^{\mu-2m'}},$$

where μ is either 2, 4, or 6.

In the following applications of these two theorems it is to be understood that the law of attraction is the inverse square.

Theorem. The potential of a spherical shell of indefinitely small thickness, whose density varies inversely as the cube of the distance from the centre at an *internal* particle, is constant and equal to $\frac{4\pi\lambda^{3}}{a'^{2}}$ where a' is its radius.

Reciprocal Theorem. The centre being the pole, the reciprocal of such a surface is a homogeneous spherical shell, and therefore, the potential of a spherical shell at an *external* point is the same as that of a certain mass collected at its centre. And this mass is $\frac{\lambda^{-3}}{\kappa^{-4}} \frac{4\pi\lambda^{3}}{a'^{2}} = 4\pi a^{2}$ which is the mass of the shell itself.

Theorem. The potential of a homogeneous spherical surface of radius a' at an internal particle Y' is constant and equal to $4\pi a'$.

Reciprocal Theorem. Take the pole S either within or without the shell, it is easy to see that the reciprocal surface of a sphere is another sphere. Therefore the potential of a spherical surface whose density varies inversely as the cube of the distance from a point S at another point Y, separated from S by the attracting surface, is the same as that of a certain mass collected at S.

This mass is $\frac{\lambda^3}{\kappa^2}\cdot 4\pi a'$. For let c be the distance of S from C the centre of the sphere A. Then S being the centre of similitude of the two spheres, if κ^2 be taken equal to the rectangle of the segments into which a chord from S is divided by either sphere, *i.e.* equal to $\pm(c^2-a^2)$ the two spheres become identical, and therefore the mass collected at $S=\frac{4\pi\lambda^3 a}{\pm(c^2-a^2)}$.

Theorem. The potential of a spherical surface of uniform density and radius a' on an external particle is the same as that of a mass $4\pi a'^2$ collected at its centre.

Reciprocal Theorem. The potential of a spherical surface, whose density varies inversely as the cube of the distance from a point S, on another point Y on the same side of the attracting surface as S, is the same as that of a certain mass collected at a point O.

This mass is $\frac{\lambda^3}{\kappa^4}\cdot SO.4\pi a'^2$, and O is given by $SO.SC=\kappa^2$, if then we take as before $\kappa^2=\pm(c^2-a^2)$ we find

$$M=\frac{4\pi\lambda^3 a}{\pm(c^2-a^2)}\,\frac{a}{c}$$

and $CO.CS=a^2$.

It may be readily shewn that the actual masses of the shells in the two last theorems are $\frac{4\pi\lambda^3 a}{a^2-c^2}$ or $\frac{4\pi\lambda^3 a}{c^2-a^2}\cdot\frac{a}{c}$ according as S is internal or external.

Theorem. The potential of a spherical shell formed by two spheres, whose radii are a', b' and the distance between whose centres $O_1'\,O_2'$ is f', at an external point Y' is the difference of the potentials of the two masses $\frac{4}{3}\pi a'^3$ and $\frac{4}{3}\pi b'^3$ collected respectively at $O_1'\,O_2'$.

*Reciprocal Theorem.** By properly choosing f' we can of course make the two reciprocal spheres concentric. Therefore, the potential of a spherical shell bounded by two concentric spheres whose radii are a, b and whose density varies in-

* When κ^2 is taken equal to c^2-a^2, a sphere and its reciprocal become identical, and the geometrical construction becomes very simple. If

versely as the fifth power of the distance from a point S at a point Y on the same side of the attracting mass as S, is the difference of the potentials of two masses collected respectively at O_1 O_2.

These two masses are respectively $\frac{\lambda^5 SO_1}{\kappa^6}\cdot\frac{4}{3}\pi a'^3$ and $\frac{\lambda^5 SO_2}{\kappa^6}\cdot\frac{4}{3}\pi b'^3$. By drawing a figure, and considering the similar triangles it will be seen that if C be the common centre,

$$SO_1 = \frac{SC^2 - a^2}{SC} \text{ and } a' = a.\frac{\kappa^2}{SC^2 - a^2},$$

$$SO_2 = \frac{SC^2 - b^2}{SC} \text{ and } b' = b.\frac{\kappa^2}{SC^2 - b^2};$$

and the masses are therefore

$$\frac{4\lambda^5\pi.a^3}{3.SC.(SC^2 - a^2)^2} \text{ and } \frac{4\lambda^5\pi b^3}{3.SC.(SC^2 - b^2)^2},$$

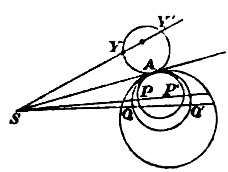
and the points O_1 O_2 may be found by

$$CO_1.CS = a^2,\ CO_2\ CS = b^2.$$

Theorem. The potential of a homogeneous spherical shell, formed as before by two eccentric spheres at an internal point Y', is equal to $2\pi(a'^2 - b'^2) - \frac{2}{3}\pi(O_1'Y'^2 - O_2'Y'^2)$.

Reciprocal Theorem. The potential of a spherical shell, formed as before by two concentric spheres and whose density varies inversely as the fifth power of the distance from a

any two spheres APP', AQQ' be described touching the given sphere



in A, where SA is a tangent, then it may be easily shown that the potential at Y of the element PQ whose density is $\frac{\lambda^5}{SP^5}$ bears a constant ratio to that at Y' of the element $P'Q'$ whose density is unity; hence the potential at Y of the whole sphere with variable density bears a known ratio to the potential at Y' of the same sphere supposed homogeneous.

point S on a point Y separated from S by the attracting mass, is

$$=\frac{2\pi\lambda^5}{\kappa^4 . SY}.(a'^2-b'^2)-\frac{2\pi\lambda^5}{3\kappa^4 . SY}(O_1'Y'^2-O_2'Y'^2)$$

$$=\frac{2\pi\lambda^5}{SY}.\left\{\frac{a^2}{(c^2-a^2)^2}-\frac{b^2}{(c^2-b^2)^2}\right\}$$

$$-\frac{2\pi\lambda^5}{3.SY}\left\{\frac{O_1Y^2}{(c^2-a^2)^2}.\frac{c^2}{SY^2}-\frac{O_2Y^2}{(c^2-b^2)^2}\frac{c^2}{SY^2}\right\},$$

$$=\frac{2\pi\lambda^5(a^2-b^2)(c^4-a^2b^2)}{(c^2-a^2)^2(c^2-b^2)^2}.\frac{1}{SY}$$

$$-\frac{2\pi\lambda^5c^2}{3}\left\{\frac{O_1Y^2}{(c^2-a^2)^2}-\frac{O_2Y^2}{(c^2-b^2)^2}\right\}.\frac{1}{SY^3}.$$

We may find the attractions of other surfaces besides spheres. For example, the potential of an ellipsoidal shell at an internal point is constant, whence taking S at the centre, the potential of the surface of elasticity, the density varying inversely as the cube of the distance from the centre at an external point, is the same as that due to its whole mass collected at its centre of gravity.

We can put these results of the two theorems in page (132) into a more convenient shape. Let N, N' be the masses of the two reciprocal bodies, then, by the same sort of reasoning as before,

$$\frac{\delta N}{\delta N'}=\frac{SP'^{m+m'-\mu}}{\kappa^{2m-\mu}}.\lambda^{m-m'};$$

$$\therefore N=\frac{\lambda^{m-m'}}{\kappa^{2m-\mu}}.\int\frac{dN'}{\overline{SP'}|^n}.$$

(1) In the first case the mass M collected at S

$$=\frac{\lambda^{m-m'}}{\kappa^{2m-\mu}}.V_0',\text{ and therefore}$$

$$\frac{M}{N}=\frac{\text{constant potential}}{\text{actual potential of whole mass at }S}.$$

If therefore we can choose Y' to coincide with S, this ratio is unity, and therefore the mass M collected at S is the actual mass of the body.

(2) In the second case, the mass M collected at O

$$= \frac{M' . \lambda^{m-m'}}{\kappa^{2m-\mu} . \overline{SO'}|^{n}} .$$

$$\therefore \frac{M}{N} = \frac{\dfrac{M'}{\overline{SO'}|^{n}}}{\displaystyle\int \frac{dN'}{(SP')^{n}}} = \frac{\text{potential at } S \text{ of } M' \text{ collected at } O'}{\text{actual potential at } S} .$$

If then, as before, we can choose Y' to coincide with S, this ratio is unity, and therefore the mass collected at O, the reciprocal of O', is the actual mass of the reciprocal body.

If the point Y' can be chosen to coincide with S, then Y can be taken at infinity, and it follows therefore that if the potential at a point Y that can be taken very distant be the same as that of a mass M collected at a point O, then M is equal to the mass of the body. It may also be shown in a similar manner that this point O is the centre of gravity of the body. The following investigation will put this in a clearer light.

Take O for origin, and let $OY = c$ and OP the distance of any particle dm at P from $O = r$, and let the angle $POY = \theta$, then

$$V = \int \frac{dm}{(c^2 - 2cr\cos\theta + r^2)^{\frac{n}{2}}}$$

$$= \int \frac{dm}{c^n} \left[1 + n\,\frac{r\cos\theta}{c} + \frac{n}{2} \left\{ (n+2)\,\frac{r^2\cos^2\theta}{c^2} - \frac{r^2}{c^2} \right\} + \dots \right] .$$

We have to determine under what circumstances this can be put into the form

$$\text{a constant} + \frac{M}{c^n} ,$$

equating the coefficients of the several powers of c, we have

$$M = \int dm ,$$

whence M is the mass of the body. Also

$$\int r\cos\theta\, dm = 0 ,$$

except when $n = -1$, and then it will be sufficient to equate it to a constant. In the first case, it follows at once that O is the centre of gravity. In the second case, since the line OY is quite arbitrary, it is easily seen that the equation cannot be satisfied unless the constant is zero, and therefore we infer the same conclusion as before. Again

$$(n+2)\int r^2 \cos^2\theta \, dm - \int r^2 dm = 0,$$

except when $n = -2$, and then it will be sufficient to equate it to a constant. Taking Y in the axis of x this becomes

$$(n+1)\int x^2 dm = \int y^2 dm + \int z^2 dm,$$

and similarly taking Y in the axes of y and z, we get

$$(n+1)\int y^2 dm = \int z^2 dm + \int x^2 dm,$$

$$(n+1)\int z^2 dm = \int x^2 dm + \int y^2 dm,$$

and these three equations cannot coexist unless $n = -2$, or $n = 1$ and the three integrals equal.

In the former case the law of attraction is as the distance, and we have, in all cases,

$$V = \int r^2 dm + c^2 \int dm$$

= constant + potential of M collected at the centre of gravity.

In the latter case, the law of attraction is as the inverse square of the distance, and the potential can only be the same as that of a mass M collected at the centre of gravity when that point is such that every axis through it is a principal axis.

In a similar manner we can determine in what cases the potential at an internal point is constant; as before

$$V = \int \frac{dm}{(c^2 - 2cr\cos\theta + r^2)^{\frac{n}{2}}}$$

$$= \int \frac{dm}{r^n}\left[1 + n\frac{c\cos\theta}{r} + \frac{n}{2}\left\{(n+2)\frac{c^2\cos^2\theta}{r^2} - \frac{c^2}{r^2}\right\} + \ldots\right],$$

expanding in powers of $\frac{c}{r}$ because c is now less than r. It may similarly be shown that this cannot be independent of c unless $n = 1$, that is, the law of attraction must be inversely as the square of the distance.

When the force of attraction varies as the simple inverse power of the distance, then $n = 0$, and the potential of an attracting mass takes a different form; but in this case the quantity we have called V becomes the *mass* of the body, and therefore the reciprocal theorems, though they no longer apply to the attractions of spheres and other bodies, will still enable us to find their masses when their density varies as some power of the distance from a point.

Theorem. The surface of a sphere of radius a' is $4\pi a'^2$ and its volume is $\frac{4}{3}\pi a'^3$.

Reciprocal Theorem. The surface of a sphere of radius a, whose density varies inversely as the fourth power of the distance from a point, is $\frac{4\pi a^2\lambda^4}{(c^2-a^2)^2}$, and the volume of a sphere, whose density varies inversely as the sixth power of the distance from an external point, is $\frac{4\pi a^3\lambda^6}{3(c^2-a^2)^3}$.

Many other theorems may be established in a similar way, but as the volume and surface of a sphere, whose density varies as any function of the distance from an external point, may be found more easily by another process, such methods are comparatively useless. For instance, by dividing the spherical surface into circular elements whose planes are perpendicular to the straight line joining Y and the centre, it may be shown that the surface of a sphere whose density varies inversely as the n^{th} power of the distance from Y, is

$$\frac{2\pi a}{c}\frac{\lambda^n}{n-2}\left\{\frac{1}{\overline{c-a}|^{n-2}}-\frac{1}{\overline{c+a}|^{n-2}}\right\},$$

or

$$\frac{2\pi a}{c}\frac{\lambda^n}{n-2}\left\{\frac{1}{\overline{a-c}|^{n-2}}-\frac{1}{\overline{a+c}|^{n-2}}\right\},$$

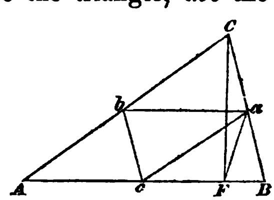
according as Y is external or internal to the sphere.

GEOMETRICAL THEOREM.

By the Rev. JOSEPH WOLSTENHOLME, Christ College.

1. THE circle passing through the middle points of the sides of a triangle passes also through the feet of the perpendiculars let fall from the angular points on the opposite sides.

For if ABC be the triangle, abc the triangle formed by



joining the middle points of its sides, and CF perpendicular to AB, join Fa: then a being the middle point of the hypothenuse of the right-angled triangle CFB, $aF = aB$,

$$\angle aFB = \angle aBF.$$

But $abcB$ is a parallelogram;

therefore $$\angle aBF = \angle abc,$$

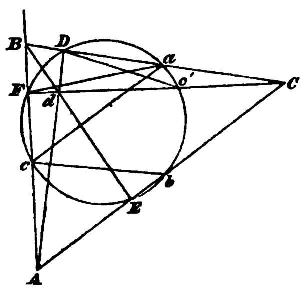
$$\angle aFB = \angle abc,$$

$$\angle abc \;\; + \angle aFc$$

equals two right angles, or the circle circumscribing abc will pass through F: and similarly through the feet of the perpendiculars from A, B, on BC, CA.

2. This circle also bisects the straight lines joining the angular points to the point of intersection of the perpendiculars from the angular points on the opposite sides.

For let AD, BE, CF the perpendiculars meet in d, and



let Cd be cut by the circle in c'; and join $c'D$, Fa. Then $\angle aDc' = \angle aFc'$ in the same segment

$$= \angle aCF,$$

because a is the middle point of the hypothenuse of the right-angled triangle CFB.

Hence $Dc' = Cc'$, and since CDd is a right angle, a circle with centre c' and radius $c'C$ or $c'D$ will meet Cc' again in d, or $c'd = c'C$, and Cd is bisected by the circle. Similarly for Ad, Bd.

Hence d is the centre of similarity of this circle and the circle circumscribing the original triangle; and any straight line drawn through d and meeting the two circles will be bisected by the inner circle.

Moreover, since A, B, C, d, are the centres of the inscribed and escribed circles of the triangle DEF, we see that the circumscribing circle of any triangle bisects all the six lines joining two and two of the centres of its inscribed and escribed circles.

The above furnishes a proof of the following problem (Senate-House, *Jan.* 4, 1855):

"Shew that the diameter of the circle, which passes through the feet of the perpendiculars from the angular points of a triangle upon the opposite sides, is equal to the radius of the circle described about the triangle."

For these two circles circumscribe similar triangles, as ABC, abc; the dimensions of one of which are twice those of the other.

A THEOREM RELATING TO SURFACES OF THE SECOND ORDER.

By A. Cayley.

Given a surface of the second order

$$(a, b, c, d, f, g, h, l, m, n) \ (x, y, z, w)^2 = 0,$$

and a fixed plane

$$\alpha x + \beta y + \gamma z + \delta w = 0,$$

imagine a variable plane

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

subjected to the condition that it always touches a surface of the second order, or what is the same thing such that the parameters ξ, η, ζ, ω satisfy a condition

$$(\mathrm{a, b, c, d, f, g, h, l, m, n}) \ (\xi, \eta, \zeta, \omega)^2 = 0.$$

The given surface of the second order, and the variable plane meet in a conic, and the fixed plane and the variable plane meet in a line, it is required to find the locus of the pole of the line with respect to the conic.

The pole in question is the point in which the variable plane is intersected by the polar of the line with respect to the surface of the second order: this polar is the line joining the pole of the fixed plane with respect to the surface of the second order, and the pole of the variable plane with respect to the surface of the second order. Let α_1, β_1, γ_1, δ_1, be given linear functions of α, β, γ, δ, and ξ, η, ζ, ω, be given linear functions of ξ, η, ζ, ω, viz., if

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}),$$

are the inverse system to $(a, b, c, d, f, g, h, l, m)$, then let

$$\begin{aligned}
\alpha_1 &= \mathfrak{A}\alpha + \mathfrak{H}\beta + \mathfrak{G}\gamma + \mathfrak{L}\delta, \\
\beta_1 &= \mathfrak{H}\alpha + \mathfrak{B}\beta + \mathfrak{F}\gamma + \mathfrak{M}\delta, \\
\gamma_1 &= \mathfrak{G}\alpha + \mathfrak{F}\beta + \mathfrak{C}\gamma + \mathfrak{N}\delta, \\
\delta_1 &= \mathfrak{L}\alpha + \mathfrak{M}\beta + \mathfrak{N}\gamma + \mathfrak{D}\delta,
\end{aligned}$$

and in like manner,

$$\begin{aligned}
\xi_1 &= \mathfrak{A}\xi + \mathfrak{H}\eta + \mathfrak{G}\zeta + \mathfrak{L}\omega, \\
\eta_1 &= \mathfrak{H}\xi + \mathfrak{B}\eta + \mathfrak{F}\zeta + \mathfrak{M}\omega, \\
\zeta_1 &= \mathfrak{G}\xi + \mathfrak{F}\eta + \mathfrak{E}\zeta + \mathfrak{N}\omega, \\
\omega_1 &= \mathfrak{L}\xi + \mathfrak{M}\eta + \mathfrak{N}\zeta + \mathfrak{D}\omega,
\end{aligned}$$

then the coordinates of the pole of the fixed plane are as

$$\alpha_1 : \beta_1 : \gamma_1 : \delta_1,$$

and the coordinates of the pole of the variable plane are as

$$\xi_1 : \eta_1 : \zeta_1 : \delta_1,$$

whence the equations of the polar are

$$\left\| \begin{matrix} x, & y, & z, & w \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \end{matrix} \right\| = 0,$$

a system of equations which may be thus represented

$$\begin{aligned}
\xi_1 &= \mathrm{K}\lambda x + \mu\alpha_1 \\
\eta_1 &= \mathrm{K}\lambda y + \mu\beta_1 \\
\zeta_1 &= \mathrm{K}\lambda z + \mu\gamma_1 \\
\omega_1 &= \mathrm{K}\lambda w + \mu\delta_1
\end{aligned}$$

where K is the discriminant of the system

$$(a, b, c, d, e, f, g, h, l, m, n),$$

write

$$\begin{aligned}
\mathrm{x} &= ax + hy + gz + lw, \\
\mathrm{y} &= hx + by + fz + mw, \\
\mathrm{z} &= gx + fy + cz + nw, \\
\mathrm{w} &= lx + my + nr + dw,
\end{aligned}$$

the last preceding system of equations may be written

$$\begin{aligned}
\xi &= \lambda\mathrm{x} + \mu\alpha, \\
\eta &= \lambda\mathrm{y} + \mu\beta, \\
\zeta &= \lambda\mathrm{z} + \mu\gamma, \\
\omega &= \lambda\mathrm{w} + \mu\delta,
\end{aligned}$$

equations in which λ, μ are indeterminate, and where $\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}$ may be considered as current coordinates, and this system represents the polar above referred to. Combining the equations in question with the equation

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

of the variable plane, we have

$$\lambda\,(x\mathrm{x} + y\mathrm{y} + z\mathrm{z} + w\mathrm{w}) + \mu\,(\alpha x + \beta y + \gamma z + \delta w) = 0,$$

i.e. $\lambda\,(a, \dots)\,(x, y, z, w)^2 + \mu\,(\alpha x + \beta y + \gamma z + \delta w) = 0,$

or what is the same thing

$$\lambda : \mu = \alpha x + \beta y + \gamma z + \delta w : (a, \dots)\,(x, y, z, w)^2,$$

and substituting these values in the expressions for ξ, η, ζ, ω we have ξ, η, ζ, ω in terms of the coordinates x, y, z, w of the pole above referred to, i.e., if for shortness,

$$U = (a, b, c, d, f, g, h, l, m, n)\,(x, y, z, w)^2,$$

$$P = \alpha x + \beta y + \gamma z + \delta w,$$

then

$$\xi = \tfrac{1}{2}Pd_x U - \alpha U,$$

$$\eta = \tfrac{1}{2}Pd_y U - \beta U,$$

$$\zeta = \tfrac{1}{2}Pd_z U - \gamma U,$$

$$\omega = \tfrac{1}{2}Pd_w U - \delta U,$$

and combining with these equations the equation.

$$(a \dots)\,(\xi, \eta, \zeta, \omega)^2 = 0,$$

we have

$$(a \dots)\,(\tfrac{1}{2}Pd_x U - \alpha U,\ \tfrac{1}{2}Pd_y U - \beta U,\ \tfrac{1}{2}Pd_z U - \gamma U,\ \tfrac{1}{2}Pd_w U - \delta U)^2 = 0,$$

for the required locus of the pole of the line of intersection of the variable plane and the fixed plane, with the conic of intersection of the given surface of the second order and the variable plane. The locus in question is a surface of the fourth order; and it may be remarked that this surface touches the given surface of the second order along the conic of intersection with the fixed plane.

7th April, 1857.

ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

By ANDREW S. HART.

I DO not pretend to add anything to the very complete and elegant *à priori* investigation of this porism given by Mr. Cayley in the 4th number of the *Quarterly Journal*, p. 344, but the following *à posteriori* demonstration of the allographic case appears to me somewhat more simple and elementary than that given in the last number of the *Journal*, pp. 35–38.

LEMMA. If two chords of a conic touch a second conic, the lines which join their extremities will touch a third conic passing through the points of intersection of the two others.

For let $A=0$, $C=0$ be the equations of the two chords, and $B=0$, $D=0$ the equations of the lines which join their extremities, then the equations of the three conics are

$$AC=BD,\quad AC=M^2,\quad BD=M^2,$$

which evidently pass through the same four points, and the four points of contact are on the same right line $M=0$.

The elementary proof of this lemma in the case of circles having the same radical axis, follows immediately from the consideration that tangents to two of these circles, from any point on the circumference of a third, bear a constant ratio to one another.

Now let ABC, abc be two triangles inscribed in the same conic P, and let AB, ab touch another conic Q, then Aa, Bb will touch a conic X passing through the intersections of P and Q, and if BC, bc* touch another conic R, passing through the same four points, then Bb, Cc will also touch the same conic X; therefore, since Aa and Cc touch this conic, AC and ac will touch a conic S which passes through the same four points. Q. E. D.

* The tangent BC being drawn, there may be two tangents drawn from b, and it is evident that only one of them will answer the conditions of the question; all ambiguity may, however be removed by observing that the right line drawn from the point of contact of Bb to the point of contact of BC must pass through the points of contact of bc and of Cc. The second tangent from b will obviously belong to the second conic which passes through the four points of intersection and touches Bb. And since this conic does not touch Aa, the remainder of the demonstration will not apply to it.

Trinity College, Dublin,
3rd April, 1857.

ON A NEW SOLVIBLE FORM OF EQUATIONS OF THE FIFTH DEGREE.

By JAMES COCKLE, M.A., F.R.A.S., F.C.P.S., Barrister-at-Law, of the Middle Temple.

IN a foot note, appended to pages 114–116 of my second paper on the method of vanishing groups, published in volume VII. of the *Cambridge and Dublin Mathematical Journal* (*May* 1852), I exemplified a process which, applied to

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0,$$

gives (in the respective cases of cubics and biquadratics) the characteristic functions

$$A^2 - 3B \text{ and } - A^3 + 4AB - 8C.$$

The same process applied to a quintic (put for convenience under the form

$$x^5 - 5Px^3 - 5Qx^2 - 5Rx + E = 0)$$

leads to a function no less characteristic; for the evanescence of one of its six values indicates that the roots are all of the form

$$\alpha X + \alpha^2 X' + \alpha^3 X'',$$

and consequently, as I have elsewhere (*Phil. Mag.*, August 1856, p. 124) pointed out, all obtainable.

The product of these six values is homogeneous and of the 24th degree with respect to the roots, and constitutes the "symmetric product" peculiar to quintics. Equated to zero it affords a new solvible form which seems to me worthy of notice for its generality, two conditions only being involved, one of which may be dispensed with by substituting $x + A$ for x in the given equation.

Let $S = P^2 + R,$

$$G = P^3Q^2 + 4(3PS - 2Q^2)PS + Q^4 + S^3 + QSE,$$

$$H = Q(6PS - Q^2) - SE,$$

$$K = P^3Q^2 + PS(PS - 6Q^2) - Q^2(PS - Q^2) + QSE,$$

and, by means of

$$u^3 - Qu^2 + P(P^2 + S)u - P^3Q = 0,$$

let u be eliminated from

$$Gu^2 + PSHu + P^2K,$$

the result, freed from extraneous factors, is the "symmetric product" of which we are in search.

Let P be zero and Q finite. The evanescence of the symmetric product depends upon

$$G = Q^4 + R^3 + QRE = 0,$$

a property of a form of Euler's.

Let P be finite and Q zero. The corresponding condition is

$$S = 0,$$

and De Moivre's form is indicated.

These are verifications of the preceding results, and tests of the extraneous factors.

There is an oversight in my solution of a biquadratic by this method. It is easily corrected, as follows.

From (see *Cambridge and Dublin Mathematical Journal*, vol. VII., p. 116)

$$P_3' = 0 \text{ and } A' = 0,$$

we deduce successively

$$y_1 = y_2, \quad y_3 = y_4, \quad B' = -(y_1^2 + y_3^2), \quad D' = (y_1 y_3)^2,$$

and the roots are, thus, determined.

76, *Cambridge Terrace, Hyde Park*,
6th April, 1857.

ON THE GEOMETRICAL INTERPRETATION OF THE EXPRESSION $rt - s^2$.

By H. W. Elphinstone, Trinity College.

(Continued from p. 76.)

It has been pointed out to me, that the condition which I gave for the existence of a ridge, in a former paper, is not sufficiently general. This may easily be seen, and the true condition found, by transformation of coordinates.

Let x, y, z be the old axes, ξ, η, ζ the the new axes, and let the cosines of the angles between them be given by the annexed table:

	ξ	η	ζ
x	a	b	c
y	a_1	b_1	c_1
z	a_2	b_2	c_2

Then if V be any function of x, y, z, we have

$$\frac{dV}{d\xi} = a\frac{dV}{dx} + a_1\frac{dV}{dy} + a_2\frac{dV}{dz},$$

$$\frac{dV}{d\eta} = b\frac{dV}{dx} + b_1\frac{dV}{dy} + b_2\frac{dV}{dz},$$

$$\frac{dV}{d\zeta} = c\frac{dV}{dx} + c_1\frac{dV}{dy} + c_2\frac{dV}{dz},$$

and if $V=0$, we have

$$\frac{dV}{d\xi} + \frac{dV}{d\zeta}\frac{d\zeta}{d\xi} = 0, \qquad \frac{dV}{d\eta} + \frac{dV}{d\eta}\frac{d\zeta}{d\eta} = 0,$$

Hence, we get

$$\frac{d\zeta}{d\xi} = -\frac{a\dfrac{dV}{dx} + a_1\dfrac{dV}{dy} + a_2\dfrac{dV}{dz}}{c\dfrac{dV}{dx} + c_1\dfrac{dV}{dy} + c_2\dfrac{dV}{dz}},$$

$$\frac{d\zeta}{d\eta} = -\frac{b\dfrac{dV}{dx} + b_1\dfrac{dV}{dy} + b_2\dfrac{dV}{dz}}{c\dfrac{dV}{dx} + c_1\dfrac{dV}{dy} + c_2\dfrac{dV}{dz}}.$$

Let $\qquad V = f(x, y) - z = 0$

be the equation to the surface; then p_1, q_1 being the new values of p and q, we shall have

$$p_1 = -\frac{ap + a_1q - a_2}{cp + c_1q - c_2},$$

$$q_1 = -\frac{bp + b_1q - b_2}{cp + c_1q - c_2}.$$

Now when there is a ridge, p and q become equal to zero, owing to a factor w, common to each of them, vanishing.

Let then $\qquad p = uw, \qquad q = vw;$

therefore
$$p_1 = -\frac{(au + a_1v)\,w - a_2}{(cu + c_1v)\,w - c_2},$$

$$q_1 = -\frac{(bu + b_1v)\,w - b_2}{(cu + c_1v)\,w - c_2}.$$

The most general forms of p and q that give rise to a ridge.

I may remark that p_1, q_1 satisfy the equation $s^2 - rt = 0$. For if $\frac{d}{dw_0}$ signifies that after differentiation w is put equal to zero, we shall have, writing x, y, z instead of ξ, η, ζ respectively,

$$r = \frac{dp_1}{dx} = \frac{d^2f}{dxdw_0} \cdot \frac{dw}{dx},$$

$$s = \frac{dp_1}{dy} = \frac{d^2f}{dxdw_0} \cdot \frac{dw}{dy}$$

$$= \frac{dq_1}{dx} = \frac{d^2f}{dydw_0} \cdot \frac{dw}{dx},$$

$$t = \frac{dq_1}{dy} = \frac{d^2f}{dydw_0} \cdot \frac{dw}{dy},$$

which values of r, s, t satisfy the equation $s^2 - rt = 0$, when $s^2 - rt = 0$ neither owing to r, s, and t being separately equal to zero, nor owing to p and q being of the above-mentioned forms.

Suppose that $s^2 - rt = 0$ without r, s, t being separately $= 0$, and without p and q being of either of the above mentioned forms.

It has been pointed out to me that I fell into an error at page 76 of the *Journal*, in supposing that because the two values of $\frac{d\xi}{d\eta}$ are in this case equal, the two branches of the curve of intersection would touch at the point. This mistake, which occurs in several elementary works, is mentioned by Mr. Salmon in his "Higher Plane Curves," p. 29. He there points out, that when the two values of $\frac{d\xi}{d\eta}$ in a plane curve become equal, there is generally a cusp and not a point of osculation. The condition for the existence of the latter is that the equation to the curve, when referred to the double point as origin, becomes of the form

$$v_1^2 + v_1 v_2 + u_n + \&c. = 0,$$

where u_n is a function of n dimensions in ξ and η.

If we apply this reasoning to the curve formed by the intersection of a surface and its tangent plane, we see that in the case which we are considering the curve of intersection will generally have a cusp; while the radius of curvature

$$R = \frac{k}{rm^2 + 2sm + t}$$

of a normal section along the tangent to the cusp will become infinite, and will change sign on each side of the cusp because $s' - rt$ does so. Hence this section will have a point of inflexion; and the form of the surface near the point may be represented as follows: Let there be a plane curve containing a point of inflexion; let a second curve in a plane perpendicular to the former contain a cusp, and move along the former so that the tangents at the cusp and at the point of inflexion may coincide. The locus of these points, which I propose to call a terrace, will separate the convex from the saddle-shaped parts of the surface. A familiar example may be seen on the common bell, the upper part of which is convex and the lower part saddle-shaped.

If on the other hand there is a true point of osculation, the radius of curvature of a normal section containing the tangent common to the two branches of the curve of intersection of the surface and its tangent plane will become infinite, but will not change sign because $s^2 - rt$ does not do so; hence the normal section will not have a point of inflexion but a point of suspended curvature. The locus of such points will, I imagine, present no peculiarity to the naked eye. We may conceive its appearance as follows: Take two bells, whose radii through the terrace are the same, cut off the convex parts and apply the two saddle-shaped parts together so that the two terraces coincide.

Should there be any difficulty in understanding how, in the one case, $s^2 - rt$ changes sign while it does not in the other, it may perhaps be removed by the following considerations.

The equation $s^2 - rt = 0$ is the equation to a curve which contains the projections of all the singular points, ridges, terraces, and loci of the last mentioned points. The plane of xy will therefore be divided into parts, in some of which $s^2 - rt$ is positive while in others it is negative. We cannot get from those parts in which it is positive, to those in which it is negative, without crossing the curve, and if we do cross the curve we must make the expression $s^2 - rt$ change sign. In the case of a terrace, when we proceed along the tangent to the cusp, we do cross the curve so that $s^2 - rt$ must change sign. In the case of the true double point of the curve of intersection, when we proceed along the tangent common to the two branches we do not cross the curve, so that in this case the expression $s^2 - rt$ cannot change sign.

ON MULTIPLE POINTS.

By H. R. Droop, M.A., Fellow of Trinity College, Cambridge.

1. The present paper contains (1) an elementary discussion according to an infinitesimal method of the principal propositions in the theory of double points, including some not to be found in Cambridge text books; (2) a translation of these propositions into language applicable to the case, that of most practical importance, where the double point is at the origin; and (3) demonstrations of several of the same propositions according to the ordinary methods of the Differential Calculus.

2. I. Let
$$\phi(x, y) = 0 \quad \text{.......................(1)},$$
where ϕ is an integral algebraic function, be the equation to a curve, and let (x, y) $(x+h, y+k)$ be points on the curve.

Then, by Taylor's theorem,
$$\phi(x+h, y+k) = 0$$
is equivalent to
$$\frac{d\phi}{dx}h + \frac{d\phi}{dy}k + \frac{1}{2}\left(\frac{d^2\phi}{dx^2}h^2 + 2\frac{d^2\phi}{dx\,dy}hk + \frac{d^2\phi}{dy^2}k^2\right)$$
$$+ \frac{1}{2.3}\left(\frac{d^3\phi}{dx^3}h^3 + 3\frac{d^3\phi}{dx^2dy}h^2k + \&c.\right) + \&c. = 0 \ldots (2).$$

3. The curve being algebraic, none of the differential coefficients of ϕ can be infinite.

It is also allowable to assume that the fact of some of the differential coefficients of a given order vanishing at a given point, while others remain finite, will not affect the essential nature of the curve at that point.

For the formulæ for the transformation of the coordinates to other rectangular axes through the same origin are
$$x = lx' + my', \quad y = mx' - ly',$$
and by successive differentiation we shall obtain
$$\frac{d^n\phi}{dx'^r dy'^{n-r}} = \Sigma A_k^{(r)} \frac{d^n\phi}{dx^{n-k}dy^k},$$
where $A_1^{(r)}$, $A_2^{(r)}$, &c. are all functions of l and m of the $\overline{n}|^{\text{th}}$ degree. If then $\dfrac{d^n\phi}{dx^{n-k}dy^k}$ be finite for any value of k, every

transformed differential coefficient $\frac{d^n\phi}{dx'^r dy'^{n-r}}$ will be finite, provided l and m be so assumed as not to make $A_k^{(r)}$ vanish for any value of r.

4. I shall sometimes use u_1, u_2, &c.... u_n for

$$\frac{du}{dx}h + \frac{du}{dy}k,$$

$$\frac{d^2u}{dx^2}h^2 + 2\frac{d^2u}{dxdy}hk + \frac{d^2u}{dy^2}k^2,$$

....

$$\frac{d^nu}{dx^n}h^n + n\frac{d^nu}{dx^{n-1}dy}h^{n-1}k + \frac{n(n-1)}{1.2}\frac{d^nu}{dx^{n-2}dy^2}h^{n-2}k^2 + \&c.,$$

according to which notation equation (2) will be written

$$u_1 + \frac{u_2}{2} + \frac{u_3}{\lfloor 3} + \&c. + \frac{u_n}{\lfloor n} + \&c. = 0.$$

5. Suppose a given small value h_1 to be given to h, the equation (2) will be very approximately satisfied by

$$k_1 = -\frac{\frac{d\phi}{dx}}{\frac{d\phi}{dy}}h_1,$$

and k_1 will be the only value of k satisfying equation (2), which is of the same order of smallness as h_1. The reader will easily see that every point $(x + h_1, y + k_1)$ will be on some branch through (x, y), while the other values of k satisfying equation (2) are finite, and correspond to the points where other branches of the curve meet the ordinate whose equation is $(\xi = x + h_1)$.

So long as $\frac{\frac{d\phi}{dx}}{\frac{d\phi}{dx}}$ has a finite and determinate value, the point (x, y) cannot be a multiple point, because k_1 has only one value for the same value of h_1; nor a conjugate point, because k_1 has a real value; nor a cusp, because it is real both for positive and for negative values of h_1.

If only one of the quantities $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ be finite, and the other be zero, the equation may, as has been shown in Art. (2), be transformed to other coordinates through the same origin, for which they will both be finite.

The point (x, y) will therefore not be either a multiple point, a conjugate point, or a cusp, unless both

$$\frac{d\phi}{dx} = 0, \text{ and } \frac{d\phi}{dy} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

6. For points where these equations are satisfied, equation (2) becomes

$$\frac{d^2\phi}{dx^2} h^2 + 2\frac{d^2\phi}{dx\,dy} hk + \frac{d^2\phi}{dy^2} k^2$$
$$+ \frac{1}{3}\left(\frac{d^3\phi}{dx^3} h^3 + 3\frac{d^3\phi}{dx^2 dy} h^2 k + 3\frac{d^3\phi}{dx\,dy^2} hk^2 + \frac{d^3\phi}{dy^3} k^3\right)$$
$$+ \frac{u_4}{3.4} + \&c. = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and if as before we assume $h = h_1$, there will be two values of k, real or imaginary, equally small with h_1. These values, and not any of the other values of k, will correspond to branches through (x, y).

7. If $$\frac{d^2\phi}{dx^2} \cdot \frac{d^2\phi}{dy^2} > \left(\frac{d^2\phi}{dx\,dy}\right)^2,$$

these two values of k will be real, for equation (4) may be transformed into

$$\left\{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)} k + \frac{\frac{d^2\phi}{dx\,dy}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}} h_1\right\}^2 = \left\{\frac{\left(\frac{d^2\phi}{dx\,dy}\right)^2}{\frac{d^2\phi}{dy^2}} - \frac{d^2\phi}{dx^2}\right\} h_1^2 - \frac{u_3}{3} - \&c.,$$

in which we may substitute for k in u_3, u_4, &c. the values obtained from the last equation by omitting those terms.

We shall thus obtain

$$\left\{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)} k + \frac{\frac{d^2\phi}{dx\,dy}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}} h_1\right\}^2 = \left\{\frac{\left(\frac{d^2\phi}{dx\,dy}\right)^2}{\frac{d^2\phi}{dy^2}} - \frac{d^2\phi}{dx^2}\right\} h_1^2 - Ph_1^3 - \&c.$$

Provided h_1 be below a certain finite magnitude, the quantity on the right-hand side of the last equation is always positive, and the two values of k are therefore real.

We shall therefore in this case have two real branches passing through the point (x, y).

If $$\frac{d^2\phi}{dx^2} \cdot \frac{d^2\phi}{dy^2} < \left(\frac{d^2\phi}{dx\,dy}\right)^2,$$

the quantity on the right-hand side of the last equation will always be imaginary, and there being no real branch through (x, y), that point is a conjugate point.

8. If $$\frac{d^2\phi}{dx^2} \cdot \frac{d^2\phi}{dy^2} = \left(\frac{d^2\phi}{dx\,dy}\right)^2,$$

the equation (4) becomes

$$\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\,h_1 + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\,k\right\}^2$$
$$= -\frac{1}{3}\left(\frac{d^3\phi}{dx^3}h_1^3 + 3\frac{d^3\phi}{dx^2dy}h_1^2k + 3\frac{d^3\phi}{dx\,dy^2}h_1k^2 + \frac{d^3\phi}{dy^3}k^3\right)$$
$$-\frac{u_4}{3.4} - \&c. \quad \ldots\ldots\ldots\ldots (5).$$

I put the positive sign before $\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}$, but of course its sign will be the same as that of $\frac{d^2\phi}{dx\,dy}$.

Substituting $k = -\dfrac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}\,h_1$, in the terms on the right-hand side of equation (5), we get

$$\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\,h_1 + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\,k\right\}^2$$
$$= -\frac{1}{3}\left\{\frac{d^3\phi}{dx^3} - 3\frac{d^3\phi}{dx^2dy}\frac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}} + 3\frac{d^3\phi}{dx\,dy^2}\left(\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}\right) - \frac{d^3\phi}{dy^3}\frac{\left(\frac{d^2\phi}{dx^2}\right)^{\frac{3}{2}}}{\left(\frac{d^2\phi}{dy^2}\right)^{\frac{3}{2}}}\right\}$$
$$+ Ph_1^4 + \&c.$$

The sign of the quantity on the right-hand side depends on that of its first term, and unless the coefficient of that term be zero, it changes with the sign of h_1. Therefore, generally, k will have two real values very nearly equal for one sign of h_1, and two imaginary values for the other, and on one side of the point (x, y) there will be two real branches with a common tangent, and on the other nothing. The point (x, y) is therefore a cusp.

9. The exceptional case in which the sign may remain the same, whether h_1 is positive or negative, is when

$$k = -\frac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}\, h_1$$

makes the coefficient of h_1^3 vanish, that is, when

$$\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\, h + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\, k$$

is a factor of u_3.

Equation (4) then becomes

$$\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\, h_1 + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\, k\right\}^2$$
$$+ \left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\, h_1 + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\, k\right\} v_2 + \frac{u_4}{3.4} + \&c. = 0 \ldots (6),$$

assuming v_2, a function of h and k of the second order,

$$= \frac{u_3}{3\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\, h + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\, k\right\}},$$

or $$\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\, h_1 + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\, k + \frac{v_2}{2}\right\}^2 = \frac{v_2^2}{4} - \frac{u_4}{3.4} - \&c.$$

Substituting $k = -\frac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}\, h_1$ as before, we shall have

$$\frac{v_2^2}{4} - \frac{u_4}{3.4} = Ph_1^4.$$

10. If P be positive, the two values of k equally small with h_1 will be real and nearly equal, whether h_1 be positive or negative. In this case there will be two real branches having a common tangent at the point, which will be what is called a point of osculation.

If P be negative, these two values of k will be imaginary, and the point will be a conjugate point; but inasmuch as the imaginary parts of these values are of the order h_1^2, the imaginary branches may be considered as having the real line $\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\,h + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\,k = 0\right\}$ through (x, y) as a common tangent.

If $P = 0$, the sign of the right-hand side will depend on the terms beyond u_4, and also on any part of v_2, which from involving the factor $\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\,h + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\,k$, was left out of P. (See the example in Art 15.)

11. The expression for P may be determined as follows:

$$v_2 = \frac{\frac{d^3\phi}{dx^3}\,h_1^3 + 3\,\frac{d^3\phi}{dx^2 dy}\,h_1^2 k + 3\,\frac{d^3\phi}{dx\,dy^2}\,h_1 k^2 + \frac{d^3\phi}{dy^3}\,k^3}{3\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\,h_1 + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\,k\right\}},$$

and when $k = -\dfrac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}\,h_1$,

the particular value of v_2 is determined by differentiating both the numerator and the denominator with respect to k_1, and is

$$\frac{3\,\frac{d^3\phi}{dx^2 dy}\,h_1^2 + 6\,\frac{d^3\phi}{dx\,dy^2}\,h_1 k + 3\,\frac{d^3\phi}{dy^3}\,k^2}{3\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}$$

$$= \frac{h_1^2}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}\left[\frac{d^3\phi}{dx^2 dy} - 2\,\frac{d^3\phi}{dx\,dy^2}\,\frac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}} + \frac{d^3\phi}{dy^3}\cdot\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}\right],$$

and therefore $P = \frac{v_2^2}{4} - \frac{u_4}{3.4}$

$$= \frac{h_1^4}{4\frac{d^2\phi}{dy^2}}\left\{\frac{d^3\phi}{dx^2dy} - 2\frac{d^3\phi}{dxdy^2}\frac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}} + \frac{d^3\phi}{dy^3}\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}\right\}^2,$$

$$- \frac{1}{3.4}\left\{\frac{d^4\phi}{dx^4}\,. - 4\frac{d^4\phi}{dx^3dy}\left(\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}\right)^{\frac{1}{2}} + 6\frac{d^4\phi}{dx^2dy^2}\left(\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}\right)\right.$$

$$\left. - 4\frac{d^4\phi}{dxdy^3}\left(\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}\right)^{\frac{3}{2}} + \frac{d^4\phi}{dy^4}\left(\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dy^2}}\right)\right\}.$$

12. If in equation (5) all the third differential coefficients of ϕ are equal to zero, the equation becomes

$$\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\,h + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\,k\right\}^2 = -\frac{u_4}{3.4} - \frac{u_5}{3.4.5} - \&c.,$$

whence $\left\{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}\,h_1 + \sqrt{\left(\frac{d^2\phi}{dy^2}\right)}\,k\right\}^2 = -Ph_1^4 - \&c.;$

and the nature of the point is determined by the sign of P, as in Art 10. It will be the same whenever the first term after u_2 is u_{2r}. But if the first term after u_2 be u_{2r+1} the whole process applied to u_3 in Articles 8—10 must be used.

13. If all the second differential coefficients of ϕ vanish the point is a triple or higher multiple point, the nature of which might be determined by a method similar to the preceding.

14. II. The general expression for an equation of the $\overline{n}|^{\text{th}}$ degree in x and y, referred to a point on the curve as origin, is

$$u_1 + u_2 + u_3 + \dots + u_n = 0, \text{ ....}(7),$$

where $u_1\ u_2 \dots u_n$ are homogeneous functions of x and y of the 1st, 2nd, ... $\overline{n}|^{\text{th}}$ degrees.

Equation (2) is of this form in h and k, and the relations between its terms, which are the conditions for the existence of different kinds of multiple points, subsist also between the corresponding terms of every equation referred to a point on the curve as origin.

From Art 5, we learn that the origin will not be either a multiple point, a conjugate point, or a cusp, if equation (7) contains the term u_1.

From Arts 6 and 7 we learn that (there being no term u_1), if u_2 divide into two unequal real factors, the origin is an ordinary double point, the two factors being the equations to the tangents there, and if u_2 divide into two imaginary factors, it is a conjugate point. From Art 8 we learn that if $u_1 = 0$, and u_2 be of the form v_1^2, the origin will be a cusp, having the line $v_1 = 0$ for the tangent there, unless u_3 is of the form $v_1 v_2$.

These three propositions are enunciated in the 2nd chapter of Mr. Salmon's *Higher Plane Curves*, pp. 27 and 28. The present paper originated in an attempt to supply an elementary demonstration of the last of them.

If equation (7) be of the form

$$v_1^2 + v_1 v_2 + u_4 + \&c. = 0,$$

we learn from Arts 9—11, that, if the results of substituting for y in terms of x from the equation $v_1 = 0$, in the expression $\frac{v_2^2}{4} - u_4$ be positive, the origin will be a point of osculation, and if negative, a conjugate point with the real line ($v_1 = 0$) for a common tangent.

If the result of the substitution be zero, the origin will generally be a cusp, but requires further examination.

15. Ex. The curve

$$a^2y^2 - ay^2x - mayx^2 + x^4 = 0,$$

has at the origin a point of osculation, a conjugate point, or a cusp, according as $m > <$ or $= 2$.

This is seen when the equation is put into the form

$$\left(ay - \frac{mx^2}{2}\right)^2 = \left(\frac{m^2}{4} - 1\right) x^4 + ay^2x,$$

the term ay^2x being left out of $v_1 v_2$, because ayx the part it would contribute to v_2 is of a higher order of smallness than $\frac{mx^2}{2}$.

16. Even when a proposed equation to a curve containing a multiple point is referred to another point as origin, it

is often easier to make the multiple point the origin and then determine the nature of the multiple point from the transformed equation, than to perform the differentiations required by the general rules given in the first part of this paper.

17. III. Mr. Salmon (*Higher Plane Curves*, p. 28) remarks that a point where two branches touch, or what I have called a point of osculation, is formed by the union of two double points. This suggests another mode of arriving at the conditions established in Art 8.

If (x, y) be one of the double points, $(x+dx, y+dy)$ will be the other. Of the equations below, those marked (2) are the conditions for (x, y) being a double point, and (8) gives the two values of $\frac{dy}{dx}$ at the point; (9) and (10) are the conditions for $(x+dx, y+dy)$ being a double point, and (11) is the equation for determining $\frac{dy}{dx}$ at the latter point, its value at which will of course be the same as at (x, y).

$$\frac{d\phi}{dx}=0, \quad \frac{d\phi}{dy}=0 \quad \ldots\ldots\ldots\ldots (2),$$

$$\frac{d^2\phi}{dx^2}+2\frac{d^2\phi}{dx\,dy}\frac{dy}{dx}+\frac{d^2\phi}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2=0 \quad \ldots\ldots\ldots\ldots (8),$$

$$\frac{d\phi}{dx}+\frac{d^2\phi}{dx^2}dx+\frac{d^2\phi}{dx\,dy}dy=0 \quad \ldots\ldots\ldots\ldots (9),$$

$$\frac{d\phi}{dy}+\frac{d^2\phi}{dx\,dy}dx+\frac{d^2\phi}{dy^2}dy=0 \quad \ldots\ldots\ldots (10),$$

$$\begin{aligned}&\frac{d^2\phi}{dx^2}+2\frac{d^2\phi}{dx\,dy}\frac{dy}{dx}+\frac{d^2\phi}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2\\ &\quad+\left\{\frac{d^3\phi}{dx^3}+2\frac{d^3\phi}{dx^2dy}\cdot\frac{dy}{dx}+\frac{d^3\phi}{dx\,dy^2}\left(\frac{dy}{dx}\right)^2\right\}dx\\ &\quad+\left\{\frac{d^3\phi}{dx^2dy}+2\frac{d^3\phi}{dx\,dy^2}\frac{dy}{dx}+\frac{d^3\phi}{dy^3}\left(\frac{dy}{dx}\right)^2\right\}dy=0 \ldots (11).\end{aligned}$$

From (2), (9), and (10), we get

$$\frac{d^2\phi}{dx^2}\cdot\frac{d^2\phi}{dy^2}=\left(\frac{d^2\phi}{dx\,dy}\right)^2,$$

a condition which Arts. 6 and 7 shew to be necessary.

Also from (9) and (11) we get

$$\frac{\frac{d^3\phi}{dx^3}+2\frac{d^3\phi}{dx^2dy}\frac{dy}{dx}+\frac{d^3\phi}{dxdy^2}\left(\frac{dy}{dx}\right)^2}{\frac{d^3\phi}{dx^2dy}+2\frac{d^3\phi}{dxdy^2}\cdot\frac{dy}{dx}+\frac{d^3\phi}{dy^3}\left(\frac{dy}{dx}\right)^2}=\frac{\frac{d^2\phi}{dx^2}}{\frac{d^2\phi}{dxdy}}=\frac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}.$$

Now if

$$\frac{d^3\phi}{dx^3}+3\frac{d^3\phi}{dx^2dy}\cdot\frac{dy}{dx}+3\frac{d^3\phi}{dxdy^2}\left(\frac{dy}{dx}\right)^2+\frac{d^3\phi}{dy^3}\left(\frac{dy}{dx}\right)^3=0\ldots(12),$$

$$\frac{dy}{dx}=-\frac{\frac{d^3\phi}{dx^3}+2\frac{d^3\phi}{dx^2dy}\frac{dy}{dx}+\frac{d^3\phi}{dxdy^2}\cdot\left(\frac{dy}{dx}\right)^2}{\frac{d^3\phi}{dx^2dy}+2\frac{d^3\phi}{dxdy^2}\left(\frac{dy}{dx}\right)+\frac{d^3\phi}{dy^3}\left(\frac{dy}{dx}\right)^2};$$

whence $\quad \dfrac{dy}{dx}=-\dfrac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}$ satisfies equation (12),

or $\sqrt{\left(\dfrac{d^2\phi}{dy^2}\right)}\cdot\dfrac{dy}{dx}+\sqrt{\left(\dfrac{d^2\phi}{dx^2}\right)}$ is a factor in its left-hand side.

This is the same condition as was deduced in Art. 8. I have shewn in Art. 10 that a point for which this condition is satisfied may be either a point of osculation, the union of two real double points, or a conjugate point with a real tangent, the union of two conjugate points, or a cusp, which may be considered either as the union of a cusp with an ordinary double point, or as its union with a conjugate point, according to the signs given to dx and dy.

18. The following equations are obtained by differentiating $\{\phi(x, y)=0\}$ regarding y as an implicit function of x,

$$\frac{d\phi}{dx}+\frac{d\phi}{dy}\frac{dy}{dx}=0\ldots\ldots\ldots\ldots(a_1),$$

$$\frac{d^2\phi}{dx^2}+2\frac{d^2\phi}{dxdy}\cdot\frac{dy}{dx}+\frac{d^2\phi}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2+\frac{d\phi}{dy}\cdot\frac{d^2y}{dx^2}=0\ldots(a_2),$$

$$\frac{d^3\phi}{dx^3}+3\frac{d^3\phi}{dx^2dy}\cdot\frac{dy}{dx}+3\frac{d^3\phi}{dxdy^2}\cdot\left(\frac{dy}{dx}\right)^2+\frac{d^3\phi}{dy^3}\left(\frac{dy}{dx}\right)^3$$
$$+3\left(\frac{d^2\phi}{dxdy}+\frac{d^2\phi}{dy^2}\cdot\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+\frac{d\phi}{dy}\frac{d^3y}{dx^3}=0\ldots(a_3),$$

$$\frac{d^4\phi}{dx^4} + 4\frac{d^4\phi}{dx^3dy}\cdot\frac{dy}{dx} + 6\frac{d^4\phi}{dx^2dy^2}\cdot\left(\frac{dy}{dx}\right)^2$$

$$+ 4\frac{d^4\phi}{dxdy^3}\cdot\left(\frac{dy}{dx}\right)^3 + \frac{d^4\phi}{dy^4}\left(\frac{dy}{dx}\right)^4$$

$$+ 6\left\{\frac{d^3\phi}{dx^2dy} + 2\frac{d^3\phi}{dxdy^2}\cdot\frac{dy}{dx} + \frac{d^3\phi}{dy^3}\cdot\left(\frac{dy}{dx}\right)^2\right\}\frac{d^2y}{dx^2} + 3\frac{d^2\phi}{dy^2}\cdot\left(\frac{d^2y}{dx^2}\right)^2$$

$$+ 4\left(\frac{d^2\phi}{dxdy} + \frac{d^2\phi}{dy^2}\cdot\frac{dy}{dx}\right)\frac{d^3y}{dx^3} + \frac{d\phi}{dy}\cdot\frac{d^4y}{dx^4} = 0 \ldots\ldots\ldots\ldots\ldots (a_4).$$

At every multiple point we have $\frac{d\phi}{dx} = 0$, and $\frac{d\phi}{dy} = 0$, and $\frac{dy}{dx}$ of the form $\frac{0}{0}$. The two values of $\frac{dy}{dx}$ at a double point are given by equation (a_2). If these values be unequal or imaginary, the corresponding values of $\frac{d^2y}{dx^2}$ may be obtained by substituting them successively for $\frac{dy}{dx}$ in equation (a_3). But when these values are equal, the coefficient of $\frac{d^2y}{dx^2}$ in equation (a_3) is zero, and that equation gives $\frac{d^2y}{dx^2} = \infty$, unless $\frac{dy}{dx} = \frac{\sqrt{\left(\frac{d^2\phi}{dx^2}\right)}}{\sqrt{\left(\frac{d^2\phi}{dy^2}\right)}}$ makes the other part of that equation $= 0$, in which case $\frac{d^2y}{dx^2}$ is of the form $\frac{0}{0}$.

The condition for $\frac{d^2y}{dx^2}$ not being $= \infty$ is the same as was found in Art. 8, for the point not being necessarily a cusp, which shews that whenever $\frac{d^2y}{dx^2} = \infty$, the point (x, y) is a cusp.

From equation (a_4) it appears that whenever $\frac{d^2y}{dx^2} = \infty$, $\frac{d^3y}{dx^3} = \infty$.

In the exceptional case two values of $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dx}$

and the differential coefficients of ϕ are given by equation (a_4). These two values will be unequal imaginary or equal according as a certain expression, which will be found to be identical with that found in Art. 11 for P, is positive, negative, or zero. This identity shews (Art. 10) that two unequal values of $\frac{d^2y}{dx^2}$ correspond to a point of osculation, and two imaginary values to a conjugate point with a real common tangent.

By differentiating (a_4) we shall get an equation (a_5), in which the coefficient of $\frac{d^3y}{dx^3}$ is

$$10\left\{\frac{d^3\phi}{dx^2dy}+2\frac{d^3\phi}{dxdy^2}\cdot\frac{dy}{dx}+\frac{d^3\phi}{dy^3}\cdot\left(\frac{dy}{dx}\right)^2+\frac{d^2\phi}{dy^2}\cdot\frac{d^2y}{dx^2}\right\},$$

a quantity which is zero, when the two values of $\frac{d^2y}{dx^2}$ derived from (a_4) are equal. This indicates that generally $\frac{d^3y}{dx^3}=\infty$, when these two values are equal; but, as before, the other part of equation (a_5) may become zero, on substituting their values as found for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, in which case $\frac{d^3y}{dx^3}$ will be of the form $\frac{0}{0}$, and two values of it will be found from the equation (a_6). If these two values of $\frac{d^3y}{dx^3}$ be real and unequal, we shall have two real branches, not only touching, but having the same curvature at the point (x, y); and if imaginary, two imaginary branches having the same curvature as any real curve, which passing through the point has $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the same value.

19. The following theorems might be deduced by examining this series of equations continued further.

(1) At an ordinary double point $\frac{dy}{dx}$ has two unequal values, and the equations (a_3), (a_4), &c.... give corresponding real values for $\frac{d^2y}{dx^2}$ $\frac{d^3y}{dx^3}$ &c.

(2) At a point of osculation of the $\overline{n}|^{\text{th}}$ order, that is, a point where two branches have a contact of the $\overline{n}|^{\text{th}}$ order, $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}$, after appearing under the form $\frac{0}{0}$ are found to have each only one value, and $\frac{d^{n+1}y}{dx^{n+1}}$ has two unequal real values.

(3) At a conjugate point either $\frac{dy}{dx}$, or some subsequent differential coefficient $\frac{d^ny}{dx^n}$ has two imaginary values. In the latter case, $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^{n-1}y}{dx^{n-1}}$, after appearing under the form $\frac{0}{0}$, are found to have each one real value, and the two imaginary branches have a contact of the $\overline{n-1}|^{\text{th}}$ order with every real curve through the point which has

$$\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^{n-1}y}{dx^{n-1}}$$

of the same values, as are finally found for them in the curve $\phi(x, y) = 0$.

(4) At a cusp some differential coefficient subsequent to $\frac{dy}{dx}$ as $\frac{d^ry}{dx^n} = \infty$, and each of the preceding differential coefficients, after appearing under the form $\frac{0}{0}$, is found to have one single real value. The two branches of the cusp will be found to have a contact of the $\overline{n-1}|^{\text{th}}$ order with each other.

NOTE ON THE 'CIRCULAR RELATION' OF PROF. MÖBIUS.

By A. Cayley.

THEOREM.

Given the points $A, B, C; P,$
and the points $A', B', C';$

Describe the circles $\alpha, \beta, \gamma, \omega$ as follows: viz.

$$\begin{array}{lll} \alpha & \text{through} & (B, C, P), \\ \beta & \text{"} & (C, A, P), \\ \gamma & \text{"} & (A, B, P), \\ \omega & \text{"} & (A, B, C), \end{array}$$

and the circles $\alpha', \beta', \gamma', \omega'$ as follows: viz.,
ω' through (A', B', C') and

α' through (B', C') cutting ω' at the angle at which α cuts ω,
β' " (C', A') " ω' " β " ω,
γ' " (A', B') " ω' " γ " ω,

then will α', β', γ' meet in a point P', i.e. we shall have the points A', B', C', P' such that the circles $\alpha', \beta', \gamma', \omega'$ pass

$$\begin{array}{lll} \alpha' & \text{passes through} & (B', C', P'), \\ \beta' & \text{"} & (C', A', P'), \\ \gamma' & \text{"} & (A', B', P'), \\ \omega' & \text{"} & (A', B', C'). \end{array}$$

We may construct in this manner two figures, such that to three points of the first figure there correspond in the second figure three points which may be taken at pleasure, but these once selected to every other point of the first figure there will correspond in the second figure a perfectly determinate point. And the two figures will be such that whenever in the first figure four or more points lie in a circle, then in the second figure the corresponding points will also lie in a circle. The relation in question is due to Prof. Möbius, who has termed it the *Kreis-verwandschaft* (circular relation) of two plane figures. See his paper *Crelle*, t. LII. pp. 218–228, extracted from the Berichten of the *Royal Saxon Society of Sciences* of the 5th Feb. 1853.

ON THE DETERMINATION OF THE VALUE OF A CERTAIN DETERMINANT.

By A. Cayley.

Considering the determinant

$$\begin{vmatrix} \theta, & 1, & . & . & \dots \\ n, & \theta, & 2 & & \\ . & n-1, & \theta, & 3 & \\ & & n-2, & \theta, & 4 \\ \vdots & & & & \end{vmatrix}$$

let the successive diagonal minors be U_0, U_1, U_2, ... U_x ..., it is easy to find

$$U_0 = 1,$$
$$U_1 = \theta,$$
$$U_2 = (\theta^2 - 1) - (n - 1),$$
$$U_3 = \theta (\theta^2 - 4) - 3 (n - 2) \theta,$$
$$U_4 = (\theta^2 - 1) (\theta^2 - 9) - 6 (n - 3) (\theta^2 - 1) + 3 (n - 3) (n - 1),$$

which in fact suggests the law, viz.

$$\begin{aligned} U_x = {} & (\theta+x-1)(\theta+x-3)(\theta+x-5)\dots(\theta-x+5)(\theta-x+3)(\theta-x+1) \\ & - \frac{x(x-1)}{2}(n-x+1)(\theta+x-3)(\theta+x-5)\dots(\theta-x+5)(\theta-x+3) \\ & + \frac{x(x-1)(x-2)(x-3)}{2.4}(n-x+1)(n-x+3)(\theta+x-5)\dots(\theta-x+5) \\ & - \&c. \\ & \vdots \\ & + (-)^s \frac{x(x-1)\dots(x-2s+1)}{2.4\dots 2s}(n-x+1)(n-x+3)\dots \\ & \quad (n-x+2s-1)(\theta+x-2s-1)(\theta+x-2s-3)\dots(\theta-x+2s+1) \\ & \vdots \end{aligned}$$

to $s = \frac{1}{2}x$ or $\frac{1}{2}(x-1)$ as x is even or odd.

And of course if x denote the number of lines or columns of the determinant, then U_x is the value of the determinant. This theorem, or a particular case of it, is due to Prof. Sylvester: I have not been able to find an easier demonstration than the following one, which, it must be admitted, is somewhat complicated. I observe that U_x satisfies the equation

$$U_x - \theta U_{x-1} + (x-1)(n-x+2) U_{x-2} = 0.$$

And writing $x-1$ and $x-2$ for x, we have the system

$$U_x - \theta U_{x-1} + (x-1)(n-x+2) U_{x-2} = 0,$$
$$U_{x-1} - \theta U_{x-2} + (x-2)(n-x+3) U_{x-3} = 0,$$
$$U_{x-2} - \theta U_{x-3} + (x-3)(n-x+4) U_{x-4} = 0,$$

or, eliminating U_{x-1} and U_{x-3},

$$U_x + \{(x-1)(n-x+2) + (x-2)(n-x+3) - \theta^2\} U_{x-2}$$
$$+ (x-2)(x-3)(n-x+3)(n-x+4) U_{x-4} = 0.$$

Suppose, for shortness,

$$(\theta+x-1)(\theta+x-3)(\theta+x-5)\ldots(\theta-x+5)(\theta-x+3)(\theta-x+1) = H_x,$$

and assume

$$U_x = A_{x,0} H_x - A_{x,1} H_{x-2} \ldots + (-)^s A_{x,s} H_{x-2s} \ldots,$$

where $A_{x,s}$ is independent of θ

U_x contains the term $(-)^s A_{x,s} \; H_{x-2s}$,

U_{x-2} contains the term $(-)^s A_{x-2,s} \; H_{x-2s-2}$,

which is to be multiplied by

$$(x-1)(n-x+2) + (x-2)(n-x+3) - \theta^2.$$

This multiplier may be written under the form

$$(x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s-1)^2$$
$$- \{\theta^2 - (x-2s-1)^2\}$$
$$= M_{x,s} - \{\theta^2 - (x-2s-1)^2\},$$

if, for shortness,

$$M_{x,s} = (x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s-1)^2.$$

Now

$$M_{x,s} - \{\theta^2 - (x-2s-1)^2\}$$

multiplied into

$$(-)^s A_{x-2,s} \; H_{x-2s-2}$$

gives rise to the terms

$$(-)^s M_{x,s} \; A_{x-2,s} \; H_{x-2s-2} - (-)^s A_{x-2,s} \; H_{x-2s},$$

(since $\{\theta^2 - (x-2s-1)^2\} \; H_{x-2s-2} = H_{x-2s}$), or, what is the same thing,

$$-(-)^s M_{x,s-1}\, A_{x-2,s-1}\, H_{x-2s} - (-)^s A_{x-2,s}\, H_{x-2s}$$
$$= -(-)^s \{M_{x,s-1}\, A_{x-2,s-1} + A_{x-2,s}\}\, H_{x-2s},$$

U_{x-4} contains the term $(-)^s A_{x-4,s}\, H_{x-2s-4}$,

or, what is the same thing, $(-)^s A_{x-4,s-2}\, H_{x-2s}$.

Hence we must have

$$A_{x,s} - (A_{x-2,s} + M_{x,s-1}\, A_{x-2,s-1})$$
$$+ (x-2)(x-3)(n-x+3)(n-x+4)\, A_{x-4,s-2} = 0,$$

where

$$M_{x,s-1} = (x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s+1)^2.$$

This may be satisfied by assuming

$$A_{x,s} = B_{x,s}\,(n-x+1)(n-x+3)\ldots(n-x+2s-1)$$

for then

$$A_{x-2,s} = B_{x-2,s}\,(n-x+3)\ldots(n-x+2s-1)(n-x+2s+1)$$
$$A_{x-2,s-1} = B_{x-2,s-1}\,(n-x+3)\ldots(n-x+2s-1)$$
$$(n-x+3)(n-x+4)\, A_{x-4,x-2}$$
$$= B_{x-4,s-2}\,(n-x+4)(n-x+3)\ldots(n-x+2s-1),$$

and consequently

$$B_{x,s}\quad (n-x+1),$$
$$-\,B_{x-2,s}\,(n-x+2s+1),$$
$$-\,M_{s,x-1}\, B_{x-2,s-1},$$
$$+\,(x-2)(x-3)(n-x+4)\, B_{x-4,s-2} = 0.$$

And if this equation be satisfied independently of n, we must have

$$B_{x,s} - B_{x-2,s} - (2x-3)\, B_{x-2,s-1} + (x-2)(x-3)\, B_{x-4,s-2} = 0,$$
$$B_{x,s} - (2s+1)\, B_{x-2,s} - \{5x-8-(x-2s+1)^2\}\, B_{x-2,s-1}$$
$$+ 4(x-2)(x-3)\, B_{x-1,s-2} = 0.$$

and these are both satisfied by

$$B_{x,s} = \frac{x.x-1\ldots x-2s+1}{2^s.1.2.3\ldots s},$$

in fact substituting this value and omitting the factor

$$\frac{(x-2)(x-3)\ldots(x-2s+1)}{2^s.1.2.3\ldots s},$$

the first equation becomes

$$x(x-1)-(x-2s)(x-2s-1)-(2x-3)\,2s+4s(s-1)=0,$$

and the second equation becomes

$$x(x-1)-(2s+1)(x-2s)(x-2s-1)$$
$$-\{5x-8-(x-2s+1)^2\}\,2s+16s(s-1)=0,$$

which are identical, the first being

$$\begin{array}{l} x^2-x \\ -x^2+(4s+1)\,x-2s(2s+1) \\ \quad -4sx \qquad +6s \\ \qquad\qquad +4s(s-1)=0, \end{array}$$

and the second being

$$\begin{array}{l} x^2-x \\ -(2s+1)\{x^2-(4s+1)\,x+2s(2s+1)\} \\ +2s\{x^2-(4s+3)\,x+(2s-1)^2+8\} \\ \qquad +16s.(s-1)=0, \end{array}$$

which may be easily verified.

Hence writing for $B_{x,s}$ its value and recapitulating, the equation

$$U_x+\{(x-1)(n-x+2)+(x-2)(n-x+3)-\theta^2\}\,U_{x-2}$$
$$+(x-2)(x-3)(n-x+3)(n-x+4)\,U_{x-4}=0$$

is satisfied by

$$U_x=A_{x,0}\,H_x-A_{x,1}\,H_{x-2}\ldots+(-)^s A_{x,s}\,H_{x-2s}\ldots$$

to $s=\frac{1}{2}x$ or $\frac{1}{2}(x-1)$ as x is even or odd, where

$$H_x=(\theta+x-1)(\theta+x-3)(\theta+x-5)\ldots(\theta-x+5)(\theta-x+3)(\theta-x+1),$$

$$A_{x,s}=\frac{x(x-1)\ldots(x-2s+1)}{2^s.1.2.3\ldots s}(n-x+1)(n-x+3)\ldots(n-x+2s-1),$$

and since for $x=0, 1, 2, 3$ the values of the expression U_x coincide with those of the first four diagonal minors, the expression gives in general the value of the diagonal minor, or when x denotes the number of lines or columns of the determinant, then the value of the determinant.

2, Stone Buildings, 1*st April*, 1857.

CORRESPONDENCE.

To the Editors of the Quarterly Journal of Pure and Applied Mathematics.

GENTLEMEN,

I wish to state that my investigations concerning Reciprocal Surfaces (to which Mr. Cayley has alluded, p. 65) have been published in Vol. XXIII. of the *Transactions of the Royal Irish Academy*. I find for the numbers which Prof. Schläfli calls A and κ the following values:

$$A = 4n(n-2)(n-3)(n^2+2n-4)$$

$$6\kappa = n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960).$$

GEORGE SALMON.

Trinity College, Dublin,
April 13, 1857.

ON THE SUMS OF CERTAIN SERIES ARISING FROM THE EQUATION $x = u + tfx$.

By A. CAYLEY.

LAGRANGE has given the following formula for the sum of the inverse n^{th} powers of the roots of the equation $x = u + tfx$,

$$\Sigma(x^{-n}) = u^{-n} + (-nu^{-n-1}fu)\frac{t}{1} + (-nu^{-n-1}f^2u)'\frac{t^2}{1.2} + \ldots (1),$$

where n is a positive integer and the series on the second side of the equation is to be continued as long as the exponent of u remains negative (*Theorie des equations numeriques*, p. 225). Applying this to the equation $x = 1 + tx^s$,

$$\Sigma(x^{-n}) = 1^{-n} - \frac{n}{1}t.1^{-n+s-1} + \frac{n(n-2s+1)}{1.2}t^2.1^{-n+2s-2}\ldots$$

$$+ (-)^q \frac{n(n-qs+q-1)\ldots(n-qs+1)}{1.2\ldots q}t^q.1^{-n+qs-q} - \&c.\ldots (2)$$

to be continued while the exponent of 1 remains negative.

Let $n = \mu s + \rho$, ρ being not greater than $s-1$, the series may always be continued up to $q = \mu$, and no further. In fact writing the above value for n and putting $q = \mu + \theta$, the general term is

$$(-)^{\mu+\theta}\frac{t^{\mu+\theta}}{1.2\ldots(\mu+\theta)}(\mu s+\rho)(\rho-\theta s+\mu+\theta-1)\ldots(\rho-\theta s+1)1^{-(\rho-\theta s+\mu+\theta)}.$$

Now if $\rho+\mu-\theta(s-1)$ is negative or zero, the term is to be rejected on account of the index of 1 not being negative, and if this quantity be positive, then since $\rho-\theta s+1$ is necessarily negative for any value of θ greater than zero, the factorial $(\rho-\theta s+\mu+\theta-1)\ldots(\rho-\theta s+1)$ begins with a positive and ends with a negative factor, and since the successive factors diminish by unity, one of them is necessarily equal to zero, or the term vanishes; hence the series is always to be continued up to $q=\mu$.

Hence

$$\Sigma(z^{-\mu s-\rho})=1-\frac{\mu s+\rho}{1}t+\frac{(\mu s+\rho)\{(\mu-2)s+\rho+1\}}{1.2}t^2\ldots$$

$$+(-)^q\frac{(\mu s+\rho)\{(\mu-q)s+\rho+q-1\}\ldots\{(\mu-q)s+\rho+1\}}{1.2\ldots q}t^q$$

$$-\&c.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Continued to $q=\mu$.

By taking the terms in a reverse order, it is easy to derive

$$(-)^\mu t^{-\mu}\Sigma(z^{-\mu s-\rho})=(\mu s+\rho)\left\{\begin{array}{l}\dfrac{(\mu+\rho-1)\ldots(\mu+1)}{2.3\ldots\rho}-\dfrac{(\mu+\rho+s-2)\ldots\mu}{2.3\ldots\rho+s}t^{-1}\\ +(-)^q\dfrac{(\mu+\rho+qs-q-1)\ldots(\mu+1-q)}{2.3\ldots(\rho+q)s}t^{-q}\\ -\&c.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).\end{array}\right.$$

Continued to $q=\mu$.

Suppose in particular $s=2$, and $t=-\dfrac{\alpha+1}{\alpha^2}$, so that the equation in x becomes $\dfrac{x-1}{x^2}=-\dfrac{\alpha+1}{\alpha^2}$, whence $x=-\alpha$ or $x=\dfrac{\alpha}{\alpha+1}$, or substituting in (2)

$$\frac{(\alpha+1)^n}{\alpha^n}+\frac{(-)^n}{\alpha^n}=1+\frac{n}{1}\frac{\alpha+1}{\alpha^2}+\frac{n(n-3)}{2}\left(\frac{\alpha+1}{\alpha^2}\right)^2+\ldots\ \ldots\ldots(5).$$

Continued to the term involving $\left(\dfrac{\alpha+1}{\alpha^2}\right)^{\frac{1}{2}n}$ or $\left(\dfrac{\alpha+1}{\alpha^2}\right)^{\frac{1}{2}(n-1)}$.

Put $\alpha=-\dfrac{a+b}{a}$; and therefore

$$\alpha+1=-\frac{b}{a},\quad \frac{\alpha+1}{\alpha}=\frac{b}{a+b},\quad \frac{\alpha+1}{\alpha^2}=\frac{ab}{(a+b)^2},$$

whence

$$\frac{a^n+b^n}{(a+b)^n}=1-\frac{n}{1}\frac{ab}{(a+b)^2}+\frac{n(n-3)}{1.2}\frac{a^2b^2}{(a+b)^4}-\&c. \;\ldots\ldots(6),$$

or $$\frac{(a+b)^n-a^n-b^n}{nab\,(a+b)}=(a+b)^{n-3}-\frac{n-3}{2}(a+b)^{n-5}ab$$

$$+\frac{(n-4)(n-5)}{2.3}(a+b)^{n-7}a^2b^2-\&c. \;\ldots\ldots\ldots(7),$$

to be continued as long as the exponent of $(a+b)$ on the second side is negative.

This formula, which is easily deducible from that for the expansion of $\cos n\theta$ in powers of $\cos\theta$, is employed by M. Stern, *Crelle*, t. xx. in proving the following theorem:

If $$S=1-\frac{n-3}{2}+\frac{(n-4)(n-5)}{2.3}-\&c. \;\ldots\ldots\ldots(8).$$

Continued to the first term that vanishes, then according as n is of the form $6k+3$, $6k\pm1$, $6k$ or $6k\pm2$,

$$S=\frac{3}{n},\quad S=0,\quad S=-\frac{1}{n},\quad S=\frac{2}{n}\;\ldots\ldots\ldots(9),$$

which is in fact immediately deduced from it by writing $b=\omega a$, ω being one of the impossible cube roots of unity. Substituting the above values of x in the equation (4),

$$(1+\alpha)^{p+1}-(1+\alpha)^{-p}$$
$$=(2p+1)\alpha\left\{1+\frac{(p+1)p}{2.3}\frac{\alpha^2}{\alpha+1}+\frac{(p+2)(p+1)p(p-1)}{2.3.4.5}\frac{\alpha^4}{(\alpha+1)^2}+\ldots\right\}$$
$$\ldots\ldots\ldots\ldots\ldots(10),$$

$$(1+\alpha)^{p}+(1+\alpha)^{-p}$$
$$=2p\left\{\frac{1}{p}+\frac{p}{2}\frac{\alpha^2}{\alpha+1}+\frac{(p+1)p(p-1)}{2.3.4}\frac{\alpha^4}{(\alpha+1)^2}+\ldots\right\}$$
$$\ldots\ldots\ldots\ldots\ldots(11),$$

whence

$$(1+\alpha)^{p+1}+(1+\alpha)^{p}=(2p+1)\,\alpha\left\{1+\frac{(p+1)p}{2.3}\frac{\alpha^2}{\alpha+1}+\ldots\right\}$$
$$+2p\left\{\frac{1}{p}+\frac{p}{2}\frac{\alpha^2}{\alpha+1}+\ldots\right\}=U \text{ suppose}\ldots\ldots(12),$$

i.e. $\Delta\,(-)^p(1+\alpha)^p=(-)^{p+1}U$ or $(1+\alpha)^p=(-)^p\,\Sigma\,(-)^{p+1}U.$

Where Δ and Σ refer to the variable p. The summation is readily effected by means of the formulæ

$$\Sigma(-)^{p+1}(2p+1)(p+s+1)\ldots(p-s)=(-)^p(p+s+1)\ldots(p-s-1),$$

$$\Sigma(-)^{p+1}(p+s)\ldots(p-s)\,2p=(-)^p(p+s)\ldots(p-s-1).$$

and thence

$$(1+\alpha)^p=\left\{1+\frac{p(p-1)}{1.2}\,\frac{\alpha^2}{1+\alpha}+\frac{(p+1)p\,(p-1)(p-2)}{1.2.3.4}\,\frac{\alpha^4}{(1+\alpha)^2}+\ldots\right\}$$
$$+\alpha\left\{\frac{p}{1}+\frac{(p+1)\,p\,(p-1)}{1.2.3.}\,\frac{\alpha^2}{1+\alpha}+\ldots\right\}\ldots\ldots(13),$$

a formula of Euler's (*Pet. Trans.* 1811) demonstrated likewise by M. Catalan (*Liouville*, t. IX., p. 161–174) by induction. It may be expressed also in the slightly different form

$$(1+\alpha)^p=\left\{1+\frac{(p+1)p}{1.2}\,\frac{\alpha^2}{1+\alpha}+\frac{(p+2)(p+1)p\,(p-1)}{1.2.3.4}\,\frac{\alpha^4}{(1+\alpha)^2}+\ldots\right\}$$
$$+\frac{\alpha}{1+\alpha}\left\{\frac{p}{1}+\frac{(p+1)p\,(p-1)}{1.2.3}\,\frac{\alpha^2}{1+\alpha}+\ldots\right\}\ldots\ldots(14).$$

The two series (13), (14) are each of them supposed to contain $p+1$ terms, p being an integer; but since the terms after these all of them vanish, the series may be continued indefinitely. Suppose the two sides expanded in powers of p, the coefficients will be separately equal, and thus the identity of the two sides will be independent of the particular values of p, or the equations (13), (14), and similarly, (10), (11), (12) are true for any values of p whatever. It is to be observed that the series for negative values of p do not differ essentially from those for the corresponding positive values; as may be seen immediately by writing $-p$ for p, and $\frac{-\alpha}{1+\alpha}$ for α.

Suppose next $s=3$, or that the equation in x is $x=1+tx^3$, to rationalise the roots of this, assume $t=\frac{4(\beta^2-1)^2}{(\beta^2+3)^3}$, then the values of x are $x=\frac{\beta^2+3}{2(\beta+1)}$, $x=-\frac{\beta^2+3}{2(\beta-1)}$, $x=\frac{\beta^2+3}{\beta^2-1}$, and hence

$$\frac{2^n\{(\beta+1)^n+(-)^n(\beta-1)^n\}+(\beta^2-1)^n}{(\beta^2+3)^r}=1-\frac{n}{1}\,t+\frac{n\,(n-5)}{1.2}\,t^2$$
$$-\frac{n\,(n-7)(n-8)}{1.2.3.}\,t^3\ldots+(-)^r\,\frac{n\,(n-2r-1)\ldots(n-3r+1)}{1.2\ldots r}\,t^r\ldots(15),$$

where $t=\frac{4(\beta^2-1)^2}{(\beta^2+3)^3}$, and the series is to be continued up to the term involving $t^{\frac{n}{3}}$, $t^{\frac{n-1}{3}}$, or $t^{\frac{n-2}{3}}$.

Again, from the formula (4) we deduce the three following forms,

$$(-)^{\mu}\frac{2^{3}\{(\beta+1)^{3\mu}+(-)^{\mu}(\beta-1)^{3\mu}\}+(\beta^{2}-1)^{3\mu}}{2^{2\mu}(\beta^{2}-1)^{2\mu}}$$

$$=3\mu\left\{\frac{1}{\mu}-\frac{(\mu+1)\mu}{2.3}t^{-1}+\frac{(\mu+3)(\mu+2)(\mu+1)\mu(\mu-1)}{2.3.4.5.6}t^{2}\right.$$

$$+\ldots(-)^{q}\frac{(\mu+2q-1)\ldots(\mu-q+1)}{2.3\ldots 3q}t^{-q}\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

$$(-)^{\mu}\frac{2^{3\mu+1}\{(\beta+1)^{3\mu+1}-(-)^{\mu}(\beta-1)^{3\mu+1}\}+(\beta^{2}-1)^{3\mu+1}}{2^{2\mu}(\beta^{2}-1)^{2\mu}(\beta^{2}+3)}$$

$$=(3\mu+1)\left\{1-\frac{(\mu+2)(\mu+1)\mu}{2.3.4}t^{-1}\right.$$

$$+\frac{(\mu+4)(\mu+3)(\mu+2)(\mu+1)\mu(\mu-1)}{2.3.4.5.6.7}t^{2}\ldots$$

$$+(-)^{q}\frac{(\mu+2q)\ldots(\mu-q+1)}{2.3\ldots 3q+1}t^{-q}\ldots\ldots\ldots(17),$$

$$(-)^{\mu}\frac{2^{3\mu+2}\{(+1)^{3\mu+2}+(-)^{\mu}(\beta-1)^{3\mu+2}\}+(\beta^{2}-1)^{3\mu+2}}{2^{2\mu}(\beta^{2}-1)^{2}\cdot(\beta^{2}+3)^{2}}$$

$$=(3\mu+2)\left\{\frac{\mu+1}{2}-\frac{(\mu+3)(\mu+2)(\mu+1)\mu}{2.3.4.5}t^{-1}\ldots\right.$$

$$+(-)^{q}\frac{(\mu+2q+1)\ldots(\mu-q+1)}{2.3\ldots(3q+2)}t^{-q}\ldots\ldots(18),$$

all of them continued up to $q-\mu$.

2, Stone Buildings,
1st *April*, 1857.

ON THE TRANSFORMATION OF COORDINATES.

By Samuel Roberts, M.A.

In a former paper (Vol. II. of this Journal, p. 39) I employed the general linear transformation from x', y', z' to $a_1x + b_1y + c_1z$, $a_2x + b_2y + c_2z$, $a_3x + b_3y + c_3z$ and pointed out the meaning of the new constants introduced. My main object now is to shew more fully than therein appeared, that this transformation is symmetrical, comprehensive, and convenient as a basis for the theory of curves.

When we transform from

$$\left.\begin{array}{l} u \text{ to } a_1x + b_1y + c_1z \\ v \text{ to } a_2x + b_2y + c_2z \\ w \text{ to } a_3x + b_3y + c_3z \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we have by inversion

$$x \text{ equivalent to } \left\{\begin{matrix} u, & v, & w \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{matrix}\right\}$$

$$y \text{ equivalent to } \left\{\begin{matrix} u, & v, & w \\ c_1, & c_2, & c_3 \\ a_1, & a_2, & a_3 \end{matrix}\right\}$$

$$z \text{ equivalent to } \left\{\begin{matrix} u, & v, & w \\ a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{matrix}\right\},$$

since we may remove the common factor

$$\left\{\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right\}.$$

Thus it is put in evidence that the three points whose coordinates are respectively proportional to $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ are the angles of the new triangle of reference (x, y, z). I may remark that the leading problem of Mr. Cayley's paper on certain forms of the equation of a conic (Vol. II. p. 45) is readily proved by the direct use of the above transformation, which enables us to deduce from a given system of conditions relative to u, v, w, the corresponding system relative to x, y, z. This is the fundamental process of which the following pages contain a few illustrations. An actual identity will be shewn to exist between several processes usually employed without reference to their common origin.

We take then, as our basis, the transformation of $U=0$ from $u\ v\ w$ to

$$a_1 \sin Ax + b_1 \sin By + c_1 \sin Cz,$$
$$a_2 \sin Ax + b_2 \sin By + c_2 \sin Cz,$$
$$a_3 \sin Ax + b_3 \sin By + c_3 \sin Cz,$$

where $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, are the coordinates of the angles A, B, C of the new triangle of reference.

The development may be written

$$\underset{a}{U}\sin^n Ax^n + \underset{ba}{\Delta}\sin^{n-1}A\sin Bx^{n-1}y + \underset{ca}{\Delta}\sin^{n-1}A\sin Cx^{n-1}z,$$

$$+\frac{1}{1.2}\left\{\underset{ba}{\Delta^2}\sin^2 By^2 + \underset{ca}{\Delta^2}\sin^2 Cz^2 + 2\underset{baca}{\Delta\Delta}\sin B\sin Cyz\right\}\sin^{n-2}Ax^{n-2},$$
$$+\ \ldots\ldots$$

$$+\underset{b}{U}\sin^n By^n + \underset{ab}{\Delta}\sin^{n-1}B\sin Ay^{n-1}x + \underset{cb}{\Delta}\sin^{n-1}B\sin Cy^{n-1}z,$$

$$+\frac{1}{1.2}\left\{\underset{ab}{\Delta^2}\sin^2 Ax^2 + \underset{cb}{\Delta^2}\sin^2 Cz^2 + 2\underset{abcb}{\Delta\Delta}\sin A\sin Cxz\right\}\sin^{n-2}By^{n-2},$$
$$+\ \ldots\ldots$$

$$+\underset{c}{U}\sin^n Cz^n + \underset{ac}{\Delta}\sin^{n-1}C\sin Az^{n-1}x + \underset{bc}{\Delta}\sin^{n-1}C\sin Bz^{n-1}y,$$

$$+\frac{1}{1.2}\left\{\underset{ac}{\Delta^2}\sin^2 Ax^2 + \underset{bc}{\Delta}\sin^2 By^2 + 2\underset{acbc}{\Delta\Delta}\sin A\sin Bxy\right\}\sin^{n-2}Cz^{n-2}.$$
$$+\ \ldots\ldots = 0, \ \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

$\underset{ba}{\Delta}$ &c., meaning $b_1D_{a_1} + b_2D_{a_2} + b_3D_{a_3}$ &c., and the subject U being suppressed. If in this equation we make $z=0$, the remaining terms give an equation in x and y, determining the intersections of U and z. Now if p be any point in AB, pa, pb, the perpendiculars from it on BC, AC, we have

$$\frac{pa\sin A}{pa\sin B} = \frac{Bp}{Ap},$$

or the ratio in which AB is divided by p. This consideration immediately conducts to the equations of the polars of the curve. They, including the tangent, are formed by successively equating to zero, the coefficients of the resulting equation when $z=0$. It is unnecessary to pursue these deductions here. If we form the discriminant of the terms in x and y, we obtain the condition that a line passing through $a_1a_2a_3$ and $b_1b_2b_3$, may touch the curve, or $a_1a_2a_3$ being variables, the equation of the tangents through $b_1b_2b_3$. As a particular case of the above transformation, if we multiply the original equation by γ^n, transform from

$$\gamma x \text{ to } \gamma x,$$

$$\gamma y \text{ to } \gamma y$$

$$\gamma z \text{ to} - \alpha x - \beta y + z',$$

and proceed to take the corresponding discriminant, we have the condition that $\alpha x + \beta y + \gamma z = 0$ may touch the curve, or the reciprocal, if $\alpha\beta\gamma$ are variables. There is, however, the usual irrelevant factor $\gamma^{n(n-1)}$, for strictly speaking we have obtained the condition that the line shall touch $\gamma^n U = 0$. It is obvious that in the foregoing process it is immaterial whether we suppose $z = 0$ before or after transformation. The former assumption is tantamount to the usual procedure.

The condition that a line passing through two given points may touch $U = 0$, may be obtained from slightly different considerations. If the first polar of a point p, the first polar of a point q, and the line through p, q, intersect in a point, (pq) must be a tangent. For let r be the point of intersection, then since r is on the first polar of p, p is on the polar line of r, and similarly q is on the polar line of r, and therefore (pq) being the polar line of a point on itself is a tangent. Hence the resultant of the elimination of xyz from

$$x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} = 0,$$

$$x''\frac{dU}{dx} + y''\frac{dU}{dy} + z''\frac{dU}{dz} = 0,$$

$$(y'z'' - y''z')\,x + (z'x'' - x'z'')\,y + (x'y'' - x''y')\,z = 0,$$

gives the required condition, containing $x'y'z'$ $x''y''z''$ in the degree $n(n-1)$ and the coefficients of U in the degree $2(n-1)$.

These equations may be put under the form

$$(x''z - x'z'')\frac{dU}{dx} + (y''z' - y'z'')\frac{dU}{dy} = 0,$$

$$(x'y'' - x''y')\frac{dU}{dx} + (y''z' - y'z'')\frac{dU}{dz} = 0,$$

$$(y'z'' - y''z')\,x + (x''z' - x'z'')\,y + (x'y'' - x''y')\,z = 0,$$

or if we require the condition that $\alpha x + \beta y + \gamma z = 0$, may touch U, we have

$$\left.\begin{aligned} \beta \frac{dU}{dx} - \alpha \frac{dU}{dy} &= 0 \\ \gamma \frac{dU}{dx} - \alpha \frac{dU}{dz} &= 0 \\ \alpha x + \beta y + \gamma z &= 0 \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a)$$

as indeed is clear from the form of a tangent. There remains however the unavoidable irrelevant factor.

If we required the condition that $V = 0$ should touch $U = 0$, and attempt to obtain it by eliminating a variable between V and U, and taking the discriminant of the result, the process is equivalent to eliminating between

$$\frac{dU}{dx} + \frac{dU}{dz} \cdot \frac{dz}{dx} = 0, \quad \frac{dV}{dx} + \frac{dV}{dz} \cdot \frac{dz}{dx} = 0,$$

$$V = 0,$$

$$\frac{dU}{dy} + \frac{dU}{dz} \cdot \frac{dz}{dy} = 0, \quad \frac{dV}{dy} + \frac{dV}{dz} \cdot \frac{dz}{dy} = 0;$$

or between

$$\frac{dU}{dx} \frac{dV}{dz} - \frac{dU}{dz} \cdot \frac{dV}{dx} = 0,$$

$$\frac{dU}{dy} \cdot \frac{dV}{dz} - \frac{dU}{dz} \cdot \frac{dV}{dy} = 0,$$

$$V = 0,$$

of which (a) is a particular case. It is easy to see, in fact, that the usual methods of forming the condition are obtained by selecting three equations from the following:

$$\frac{dU}{dx} \frac{dV}{dy} + \frac{dU}{dy} \frac{dV}{dx} = 0, \ldots\ldots\ldots\ldots\ldots\ldots \; U = 0,$$

$$\frac{dU}{dx} \frac{dV}{dz} - \frac{dU}{dz} \frac{dV}{dx} = 0, \ldots\ldots\ldots\ldots\ldots\ldots \; V = 0,$$

$$\frac{dU}{dy} \cdot \frac{dV}{dz} - \frac{dU}{dz} \cdot \frac{dV}{dy} = 0, \ldots\ldots\ldots\ldots\ldots\ldots \; (3),$$

the simplest system, as Mr. Salmon has shewn, being

$$U = 0,$$

$$V = 0,$$

$$\frac{dU}{dx} \cdot \frac{dV}{dy} - \frac{dU}{dy} \cdot \frac{dV}{dx} = 0;$$

wherein it is easy to account for the irrelevant factor.

If each of the coefficients of the terms in x and y of (2) vanishes, U will contain the linear factor z, or a line passing through $a_1a_2a_3$ and $b_1b_2b_3$. We see then that in this case any point A on the factor line is on all the polars of any other point B on that line, and *vice versâ*. In fact the series of conditions implies a tangent having a contact of the $n+1^{\text{th}}$ degree, which can only be the case when z is a factor, for it can usually meet the curve in only n points. The conditions that U may contain $\alpha x+\beta y+\gamma z=0$, will be obtained more simply by a particular transformation. Multiply the original curve by γ^n, and transform from

$$\gamma x \text{ to } \gamma x,$$
$$\gamma y \text{ to } \gamma y,$$
$$\gamma z \text{ to } -\alpha x-\beta y+z',$$

we have then $(n+1)$ equations of condition, two of which contain $\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}$, uniquely. There are therefore $(n-1)$ independent conditions, (as is indeed evident from other considerations) that a curve of the n^{th} degree should have a linear factor. I do not however see how the conditions obtained from these equations by the elimination of $\alpha\beta\gamma$ are to be reduced to their lowest terms in the coefficients. In the case of the second degree, we have

$$F\alpha^2-2D\alpha\gamma+A\gamma^2=0,$$
$$F\beta^2-2E\beta\gamma+C\gamma^2=0,$$
$$F\alpha\beta-D\beta\gamma-E\alpha\gamma+B\gamma^2=0,$$

which give the square of the condition with a factor F^2. The geometrical meaning of these equations and those of higher curves readily appears.

If we write the terms of (2) in x and y in the form

$$\Pi\,(xy_1+yx_1)\,(xy_2+yx_2)\ldots\ldots(xy_n+yx_n)=0,$$

$x_1\,x_2\ldots y_1\,y_2$ being the roots, we see that the series of conditions is given by making this equation true independently of x and y, and we have a form equivalent to that obtained by Mr. Cayley's method, viz., by eliminating $x\,y\,z$ from

$$U=0,$$
$$Z=0,$$
$$\alpha u+\beta v+\gamma w=0,$$

where Z is a linear factor. In this general form, however, too many conditions are given. The conditions express that a root $\frac{x_k}{y_k}$ becomes indeterminate, but although this indeterminateness may be denoted by $\frac{0}{0}$ it does not indicate that $x_k = 0$, and $y_k = 0$ absolutely, which would imply that x, y, z' meet in a point. The transformation is, as I have observed, tantamount in effect to substituting in $\gamma^n u\,(xyz) = 0$, the value of γz derived from $ax + \beta y + \gamma z = 0$. In fact, if a curve $U = 0$ contains a curve $V = 0$, then the conditions complied with, are expressed by substituting in U the values of z in terms of x and y derived from $V = 0$, and equating the respective coefficients of the result to zero. If these conditions are satisfied, *à fortiori*, they will be more than sufficient for the case in which U and V contain a common factor of a lower degree.

We obtain in this way an equation $\phi\,(x, y) = 0$, and consequently, the conditions are given by

$$\Pi\,(xy_1 + x_1 y)\,(xy_2 + xy_2) \ldots (xy_{nm} + xy_{nm}) = 0,$$

independently of x and y, m and n being the degrees of U and V. We have therefore again, in the general case, a form equivalent to making the resultant of

$$U = 0,$$

$$V = 0,$$

$$ax + \beta y = 0,$$

equal to zero independently of a and β. Thus appears the identity of results obtained by considering that an arbitrary line through any point must meet the curves U and V in a common point, or considering that one or more of the values of z derived from $V = 0$ must satisfy $U = 0$, independently of x and y.

In the application, however, to the theory of curves, we are chiefly concerned with the case in which U contains a given linear factor, and simple transformation gives us the requisite conditions. For instance, to determine the Hessian, we have

$$\underset{x_1 x}{\Delta} \text{ a factor in } \underset{x_1 x}{\Delta^2},$$

or $Lx_1 + My_1 + Nz_1$ a factor in

$$Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + 2Dx_1z_1 + 2Ey_1z_1 + 2Fx_1y_1 = 0,$$

and transforming

$$x_1 \text{ to } x_1,$$

$$y_1 \text{ to } -\frac{L}{M}x_1 - \frac{N}{M}z_1 + y',$$

$$z_1 \text{ to } z_1,$$

we have

$$\phi(M, -L, O) = 0,$$

$$\phi(O, -N, M) = 0,$$

$$M, -L, O \Delta O, -N, M = 0,$$

and

$$AM^2 - 2BLM + CL^2 = \frac{n}{n-1}(AC - B^2)U - \frac{1}{(n-1)^2}Hz^2 = 0,$$

$$FM^2 - 2ENM + CN^2 = \frac{n}{n-1}(FC - E^2)U - \frac{1}{(n-1)^2}Hx^2 = 0,$$

$$(EL - DM)M + (BM - CL)N$$

$$= \frac{n}{n-1}(DC - EB)U - \frac{1}{(n-1)^2}Hxz = 0.$$

If (xyz) be on the curve

$$H = 0,$$

for $x = 0$, $z = 0$ are inadmissible as implying that y', x, z intersect in the point of contact.

In like manner, in the case of contact of a higher order, we must have

$$\underset{x_1 x}{\Delta} \text{ a factor in } \underset{xx}{\Delta}^k, \text{ or } \phi;$$

and proceeding, as before, we should have $k+1$ conditions, only one of which it is necessary to consider; for instance,

$$\phi(M, -L, O) = 0.$$

If (xyz) is on the curve, this will reduce to

$$Q_k z^2 = 0,$$

and, by means of other equations of the system, we shall find

$$Q_k x^2 = 0,$$

$$Q_k xz = 0;$$

so that $Q_k = 0$ will be the condition required for a contact of the $k+1^{\text{th}}$ order. The problem of finding the double tangents whose points of contact are distinct involves similar considerations. Let $\phi=0$ be the condition that

$$\lambda^{n-2}\underset{x_1x}{\Delta^2}+\lambda^{n-3}\mu\underset{x_1x}{\Delta^3}+\ldots+\mu^{n-2}U=0$$

shall have equal roots, then ϕ must contain

$$\underset{x_1x}{\Delta} \text{ or } Lx_1+My_1+Nz_1;$$

and we shall have a series of equations

$$\phi\,(M,-L,O)=0,$$
$$\phi\,(O,-N,M)=0,$$
&c.

any one of which will be sufficient to give the required condition, for $\underset{x_1x}{\Delta}$ intersects ϕ in only one point. Since the series of equations is obtained by eliminating z_1 between $\underset{x_1x}{\Delta}=0$, and $\phi=0$, and ϕ is of the degree $(n+2)(n-3)$, we shall have

$$\Pi\,(xy_1+x_1y)^{(n-2)(n-3)}=0,$$

and $\Pi=0$ will be the condition sought, the degree of which in the coefficients, and in $x\,y\,z$, is easily determined.

The number of double tangents, however, can be determined by a different mode. On examining the coefficients of the developed equation, we find, if z is a double tangent,

$$\phi\,(a_1a_2a_3)=0,$$
$$b_1\frac{du}{da_1}+b_2\frac{du}{da_2}+b_3\frac{du}{da_3}=0,$$
$$a_1\frac{du}{db_1}+a_2\frac{du}{db_2}+a_3\frac{du}{db_3}=0,$$
$$\phi\,(b_1b_2b_3)=0.$$

Eliminating from the three last $b_1b_2b_3$, we obtain an equation of the degree $n(n-1)^2+n$ or n^3-2n^2+2n in $a_1a_2a_3$, being the locus of points, whose polar lines and first polars intersect on the curve. The locus must contain the square of the curve.

This gives $n\,(n^3-2n^2)$ points of intersection with $\phi\,(a_1a_2a_3)=0$, where the tangents are double, but it includes three times the number of points of inflexion, which are points of contact of double tangents touching consecutively. Hence we have

$$n\,\{n^3-2n^2-3\,.\,3\,(n-2)\},$$

for the number of contacts of double tangents whose points of contact are distinct, or

$$\frac{n(n+3)(n-2)(n-3)}{2},$$

for the number of double tangents. Possibly a similar method might be advantageously employed in the case of surfaces.

With regard to multiple contacts not distinct, the conditions appear to take a simpler form, if we assume the satisfaction of those prior in order to the one we are dealing with. We have seen that the conditions for a contact of the k^{th} order implies that $a_1a_2a_3$ any point on the line of contact lies on the curve, and on the $n-1^{\text{th}}$ $n-2^{\text{th}}$ ... $(n-k+1)^{\text{th}}$ polars of $b_1b_2b_3$ any other point on the line. Now the line must evidently take the form

$$x\frac{du}{da_1}+y\frac{du}{da_2}+z\frac{du}{da_3}=0,$$

and we may consider it as passing through $\frac{du}{da_2}, -\frac{du}{da_1}$, 0 and $a_1a_2a_3$. Therefore we have

$$U_a=0,$$

$$\frac{du}{da_2}\frac{du}{da_1}-\frac{du}{da_1}\cdot\frac{du}{da_2}=0, \text{ satisfied identically,}$$

$$\frac{du^2}{da_2}\frac{d^2u}{da_1}-2\frac{du}{da_1}\cdot\frac{du}{da_2}\cdot\frac{d^2u}{da_1\,da_2}+\frac{du^2}{da_1{}^2}\frac{d^2u}{da_2{}^2}=0,$$

&c. &c.

and writing $\frac{du}{da_1}=L$, $\frac{du}{da_2}=M$, the law of formation is

$$M\left(\frac{d\Delta^p}{da_1}\right)-L\left(\frac{d\Delta^p}{da_2}\right)=\Delta^{p+1},$$

the differentiation being taken on the supposition that M and L are constant. From this we obtain (Salmon's *H. P. C.*, p. 82),

$$\Delta^{p+1}=M\frac{d\Delta}{da_1}-L\frac{d\Delta}{da_2}+\frac{p(n-p+1)}{n-1}F\Delta^{p-1}$$

$$-\frac{kz}{n-1}\left(D\frac{d\Delta^{p-1}}{da_1}+E\frac{d\Delta^{p-1}}{da_2}+F\frac{d\Delta^{p-1}}{da_3}\right).$$

But if U and Δ^{p-1} touch at $a_1a_2a_3$, we have

$$L\frac{d\Delta^{p-1}}{da_2}-M\frac{d\Delta^{p-1}}{da_1}=0,$$

$$L\frac{d\Delta^{p-1}}{da_3}-N\frac{d\Delta^{p-1}}{da_1}=0,\quad \Delta^{p-1}=0,$$

$$M\frac{d\Delta^{p-1}}{da_3}-N\frac{d\Delta^{p-1}}{da_2}=0,$$

and consequently, if the third condition above given be satisfied, that is, if $H=0$, we have

$$\Delta^{p+1}=M\frac{d\Delta^{p}}{da_1}-L\frac{d\Delta^{p}}{da_2},$$

the differentiation being complete. And instead of differentiating Δ^p, we may employ the form to which Δ^p reduces by the aid of the prior conditions, for since $\Delta^p=0$, and Δ^{p-1}, Δ^{p-2}, &c., touch U and one another at $a_1a_2a_3$, it will be sufficient in order that Δ^p may touch U, at the same point, that the reduced form of Δ^p touch U.

The factor z^2 being omitted, and H substituted for the third condition, we have then

$$U=0,$$

$$H=0,$$

$$\frac{dU}{da_1}\frac{dH}{da_2}-\frac{dU}{da_2}\frac{dH}{da_2}=0,$$

....

$$\left(\frac{dU}{da_1}\frac{d}{da_2}-\frac{dU}{da_2}\frac{d}{da_1}\right)^{k-3}H=0.$$

These conditions imply that U and H have a contact of the $(k-2)^{\text{th}}$ order. It is also evident that we may treat either U or H as the subject of the operation, and may consequently write

$$U=0,$$

$$H=0,$$

$$\frac{dH}{da_1}\cdot\frac{dU}{da_2}-\frac{dH}{da_2}\cdot\frac{dU}{da_1}=0,$$

....

$$\left(\frac{dH}{da_1}\frac{d}{da_2}-\frac{dH}{da_2}\frac{d}{da_1}\right)^{k-3}U=0,$$

which, however, are higher in degree. The above conditions that two curves may have a multiple consecutive contact, have a close analogy to those necessary, that a right line may have a similar contact with a curve. They, however, of course give rise to irrelevant factors. If then U and H have a contact of the k^{th} order, U has a tangent at that point of the $(k+2)^{\text{th}}$ order. This indeed appears from the fact that at every intersection of U and H, the tangent has a contact of the 3^{rd} order, and if the tangent touch at 1234, it may be considered to pass through 123, 234 if at 12345 it passes through 123, 234, 345, and so on.

The development (2) is convenient for shewing at a glance the meaning of the disappearance of any particular coefficient. For instance, suppose we require the signification of

$$\text{the coefficient of } x^m y^n z^p = 0.$$

This will be equivalent to

$$\underset{ac}{\Delta^m}\underset{bc}{\Delta^n}.\underset{c}{U} = 0,$$

and indicates that C is on the m^{th} polar of A, with regard to the n^{th} polar of B, with regard to the curve. The other relations are determined by symmetrical transposition. The coefficient of $x^n y^n$, where $2n$ is the degree of the curve, is $\underset{ac}{\Delta^n}\underset{bc}{\Delta^n}$, or $\underset{ab}{\Delta^n}\underset{b}{u}$, or $\underset{ba}{\Delta^n}\underset{a}{u}$, and if it vanishes we see that A is on the n^{th} polar of B, and *vice versâ*. Thus, if the coefficient of x^2y^2, y^2z^2. x^2z^2 in a curve of the 4^{th} degree vanish, the triangle of reference has a kind of self-conjugate property, the second polar of one angle passing through the other two. Again, if the coefficient of $x^n y^n z^n = 0$, A is on the n^{th} polar of B, with regard to the n^{th} polar of C.

An analogous method of transformation is applicable to tridimensional space. If the plane

$$\alpha = x' \cos\alpha + y' \cos\beta + z' \cos\gamma - p' = 0$$

be transformed into the shape

$$lx + my + nz + p\omega = 0,$$

where

$$x = x' \cos\alpha_1 + y' \cos\beta_1 + z' \cos\gamma_1 - p_1 \text{ ....}(a),$$

$$y = x' \cos\alpha_2 + y' \cos\beta_2 + z' \cos\gamma_2 - p_2 \text{ ....}(b),$$

$$z = x' \cos\alpha_3 + y' \cos\beta_3 + z' \cos\gamma_3 - p_3 \text{ ....}(c),$$

$$\omega = x' \cos\alpha_4 + y' \cos\beta_4 + z' \cos\gamma_4 - p_4 \text{ ....}(d),$$

we shall have four equations to determine l, m, n, p. Elimination gives

$$l = \begin{Bmatrix} \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \\ \cos\alpha_3, & \cos\beta_3, & \cos\gamma_3 \\ \cos\alpha_4, & \cos\beta_4, & \cos\gamma_4 \end{Bmatrix} a_1 \div \begin{Bmatrix} \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1, & p_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2, & p_2 \\ \cos\alpha_3, & \cos\beta_3, & \cos\gamma_3, & p_3 \\ \cos\alpha_4, & \cos\beta_4, & \cos\gamma_4, & p_4 \end{Bmatrix}.$$

Where a_1 is the perpendicular from the intersection of (b), (c), (d) on a. Now if we take a line

$$\cos\gamma_2 x = \cos\alpha_2 z,$$

$$\cos\gamma_2 y = \cos\beta_2 z,$$

perpendicular to (b) and a line parallel to the intersection of (c) and (d) in the form

$$(\cos\alpha_3 \ \cos\beta_4)' x = (\cos\gamma_4 \ \cos\beta_3)' z,$$

$$(\cos\beta_3 \ \cos\alpha_4)' y = (\cos\gamma_4 \ \cos\alpha_3)' z,$$

$\{$where $(\cos\alpha_3 \cos\beta_4)' = \cos\alpha_3 \cos\beta_4 - \cos\beta_3 \cos\alpha_4\}$ the angle between these lines will be given by

$$\cos\theta' = \begin{Bmatrix} \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \\ \cos\alpha_3, & \cos\beta_3, & \cos\gamma_3 \\ \cos\alpha_4, & \cos\beta_4, & \cos\gamma_4 \end{Bmatrix}$$

$$\div \{(\cos\alpha_3 \ \cos\beta_4)'^2 + (\cos\gamma_4 \ \cos\beta_3)'^2 + (\cos\gamma_4 \ \cos\alpha_3)'^2\}^{\frac{1}{2}}.$$

The divisor is equivalent to

$$\{1 - (\cos\alpha_3 \cos\alpha_4 + \cos\beta_3 \cos\beta_4 + \cos\gamma_3 \cos\gamma_4)^2\}^{\frac{1}{2}},$$

or to $\sin\phi$, ϕ being the angle between the planes (c) and (d), and the value of the determinant

$$\begin{Bmatrix} \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \\ \cos\alpha_3, & \cos\beta_3, & \cos\gamma_3 \\ \cos\alpha_4, & \cos\beta_4, & \cos\gamma_4 \end{Bmatrix},$$

is $\sin\phi \sin\theta$, θ being the complement of θ' or the angle made by the intersection of (c) and (d) with the plane (b).

Dividing out then this major determinant as a factor common to l, m, n, p, we may write

$$l \text{ equivalent to } \sin Ab \sin cd \,.\, a_1,$$

Ab meaning the angle made by the edge of the new tetrahedron passing through A the intersection of (b), (c), and (d), with the plane (b) and cd denoting the angle between (c) and (d); we shall also have, using a similar notation,

$$m \text{ equivalent to } \sin Bc \,.\, \sin ad \,.\, \alpha_2,$$

$$n \text{ equivalent to } \sin Ca \,.\, \sin bd \,.\, \alpha_3,$$

$$p \text{ equivalent to } \sin Dc \,.\, \sin ac \,.\, \alpha_4,$$

α_2, α_3, α_4 being the perpendiculars from B, C, D, on α.

And if we take three other planes β, γ, δ, and transform in a similar manner, we shall obtain analogous results relative to these planes, and (a) (b) (c) (d) taken in corresponding threes. That is to say, if we transform the equation

$$\phi(\alpha\beta\gamma\delta) = 0,$$

from $\alpha\beta\gamma\delta$ as planes of reference to

$$l_1x + m_1y + n_1z + p_1\omega = 0,$$
$$l_2x + m_2y + n_2z + p_2\omega = 0,$$
$$l_3x + m_3y + n_3z + p_3\omega = 0,$$
$$l_4x + m_4y + n_4z + p_4\omega = 0,$$

then l_1, l_2, l_3, l_4 will be the coordinates relative to $\alpha\beta\gamma\delta$, of $(y\,z\,\omega)$ multiplied by $\sin Ab \,.\, \sin cd$, m_1, m_2, m_3, m_4 will be the similar coordinates of $(x\,z\,\omega)$ multiplied by $\sin Bc \,.\, \sin ab$, n_1, n_2, n_3, n_4 will be the similar coordinates of $(x\,y\,\omega)$ multiplied by $\sin Ca \,.\, \sin bd$, and p_1, p_2, p_3, p_4 will be the similar coordinates of $(x\,y\,z)$ multiplied by $\sin Db \,.\, \sin ac$.

The consideration of right-angled spherical triangles at the angles of the tetrahedron of reference gives

$$\sin Ab \;\sin cd = \sin Ac \;\sin bd = \sin Ad \;\sin bc,$$

$$\sin Ba \;\sin cd = \sin Bc \;\sin ad = \sin Bd \;\sin ac,$$

$$\sin Ca \;\sin bd = \sin Cb \;\sin ad = \sin Cd \;\sin ab,$$

$$\sin Da \;\sin bc = \sin Db \;\sin ac = \sin Dc \;\sin ab.$$

In making the transformation, it is convenient to suppose the new quantities $xyz\omega$ to contain the above factors implicitly. The development is easily obtained by working out

$$\phi(a_1+b_1+c_1+d_1,\; a_2+b_2+c_2+d_2,\; a_3+b_3+c_3+d_3,\; a_4+b_4+c_4+d_4),$$

which gives the successive coefficients, while the corresponding powers of the variables are determined by the number of

No. 7. February, 1858.

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστημὴ πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

JOHN W. PARKER AND SON, WEST STRAND.

1858.

W. METCALFE, PRINTER, GREEN STREET, CAMBRIDGE.

PRICE FIVE SHILLINGS.

CONTENTS.

NOTICES TO CORRESPONDENTS.

We hope to insert the Rev. P. FROST's Paper on the "Planetary Theory," and the Rev. H. HOLDITCH's Paper on the "n^{th} Caustic," in the next Number.

Papers for the Journal and other communications may be addressed to the Editors under cover to Messrs. J. W. PARKER & SON, 445, *West Strand, London;* N. M. FERRERS, Esq., *Gonville and Caius College, Cambridge; or to the Printing Office, Green Street, Cambridge.*

The next Number, which will complete the Second Volume, will appear in May.

times the different letters enter. a_1, &c., a_2, &c., a_3, &c., a_4, &c., will be on the above assumption the coordinates of the angles of the new tetrahedron with respect to the old. We have then

$$\begin{aligned}
&\underset{a}{U}x^n + \underset{ba}{\Delta}yx^{n-1} + \underset{ca}{\Delta}zx^{n-1} + \underset{da}{\Delta}\omega x^{n-1} \\
&\quad + \frac{1}{1.2}\{\underset{ba}{\Delta^2}y^2 + \underset{ca}{\Delta^2}z^2 + \underset{da}{\Delta^2}\omega^2 + 2\underset{ba\,ca}{\Delta\Delta}yz + 2\underset{ba\,da}{\Delta\Delta}y\omega + 2\underset{ca\,da}{\Delta\Delta}z\omega\}x^{n-2} \\
&\qquad + \ldots\ldots \\
&+\underset{b}{U}y^n + \underset{ab}{\Delta}xy^{n-1} + \underset{cb}{\Delta}zy^{n-1} + \underset{db}{\Delta}\omega y^{n-1} \\
&\quad + \frac{1}{1.2}\{\underset{ab}{\Delta^2}x^2 + \underset{cb}{\Delta^2}z^2 + \underset{db}{\Delta^2}\omega^2 + 2\underset{ab\,cb}{\Delta\Delta}xz + 2\underset{ab\,db}{\Delta\Delta}x\omega + 2\underset{cb\,db}{\Delta\Delta}z\omega\}y^{n-2} \\
&\qquad + \ldots\ldots \\
&+\underset{c}{U}z^n + \underset{ac}{\Delta}xz^{n-1} + \underset{bc}{\Delta}yz^{n-1} + \underset{dc}{\Delta}\omega z^{n-1} \\
&\quad + \frac{1}{1.2}\{\underset{ac}{\Delta^2}x^2 + \underset{bc}{\Delta^2}y^2 + \underset{dc}{\Delta^2}\omega^2 + 2\underset{ac\,bc}{\Delta\Delta}xy + 2\underset{ac\,dc}{\Delta\Delta}x\omega + 2\underset{bc\,dc}{\Delta\Delta}y\omega\}z^{n-2} \\
&\qquad + \ldots\ldots \\
&+\underset{d}{U}\omega^n + \underset{ad}{\Delta}x\omega^{n-1} + \underset{bd}{\Delta}y\omega^{n-1} + \underset{cd}{\Delta}z\omega^{n-1} \\
&\quad + \frac{1}{1.2}\{\underset{ad}{\Delta^2}x^2 + \underset{bd}{\Delta^2}y^2 + \underset{cd}{\Delta^2}z^2 + 2\underset{ad\,bd}{\Delta\Delta}xy + 2\underset{ad\,cd}{\Delta\Delta}xz + 2\underset{bd\,cd}{\Delta\Delta}yz\}\omega^{n-2} \\
&\qquad + \ldots\ldots = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).
\end{aligned}$$

The coefficients satisfy the identity

$$\frac{1}{1.2\ldots m+n}\underset{ac}{\Delta^m}\underset{bc}{\Delta^n}\underset{c}{U} = \frac{1}{1.2\ldots m+p}\underset{ab}{\Delta^m}\underset{bc}{\Delta^p}\underset{b}{U},$$

$m+n+p$ being the degree of the curve. This equality expresses that if C is on the m^{th} polar surface of A, with regard to the n^{th} polar surface of B with regard to the surface U, then B is on the m^{th} polar surface of A with regard to the p^{th} polar surface of C with regard to U; and similar interpretations may be made of the other symmetrical forms of the coefficients. We thus see, as in the analogous case of plane curves, what is implied by the vanishing of a given coefficient.

To obtain the equation of the curve of intersection with the surface made by a plane passing through three given points a_1, &c., b_1, &c., c_1, &c., we have only to make $\omega = 0$ in (3).

Equating the coefficients to nothing, we obtain the conditions that the surface U may contain ω a plane passing through those three given points. If ω is a tangent, the equation in xyz thus obtained must have two equal sets of roots; that is, the curve of section, as Mr. Cayley has remarked, must have a double point. This is readily seen by projecting the coordinates x, y, z on ω. If ω is a tangent at a_1, &c., we must have

$$\underset{a}{U} = 0 = \underset{aa}{\Delta},$$

$$\underset{ba}{\Delta} = 0,$$

$$\underset{ca}{\Delta} = 0,$$

or

$$\underset{xa}{\Delta} = 0$$

is the equation of the tangent plane at a_1, &c. If we take the determinant of the terms in x, y, z, and consider one of the sets of coordinates as variable, we have the equation of tangent planes through two given points.

To obtain the condition that a plane

$$lx + my + nz + p\omega = 0$$

should touch, we must multiply the original equation (in $xyz\omega$ suppose) by p^n and transform from

$$px \text{ to } px,$$

$$py \text{ to } py,$$

$$pz \text{ to } pz,$$

$$p\omega \text{ to } -lx - my - nz + \omega.$$

It is clear this process is the same as substituting the value of ω derived from the equation of the plane in that of the surface.

If we require the equation giving the intersections of a line formed by the intersection of z and ω, we must make these variables vanish in (3); we thus obtain

$$\left.\begin{array}{l} \underset{a}{U}x^n + \underset{ba}{\Delta}yx^{n-1} + \frac{1}{2}\underset{ba}{\Delta^2}y^2x^{n-2} + \dots \\ + \underset{b}{U}y^n + \underset{ab}{\Delta}xy^{n-1} + \frac{1}{2}\underset{ab}{\Delta^2}x^2y^{n-2} + \dots \end{array}\right\} = 0 \;\dots\dots\; (4).$$

Let R be any point on the edge AB of the tetrahedron $ABCD$, and Ra, Rb, 0, 0 its coordinates, then we have

$$\sin Ab \,.\, AR = Rb,$$

$$\sin Ba \,.\, BR = Ra,$$

$$\frac{BR}{AR} = \frac{Ra \sin Ab}{Rb \sin Ba} = \frac{x}{y}.$$

Hence, the coefficients of (4) equated successively to 0 give the equations of the successive polar surfaces. (4) is in fact equivalent to the usual substitution for $\alpha\beta\gamma\delta$ of

$$a_1x + b_1y, \quad a_2x + b_2y, \quad a_3x + b_3y, \quad a_4x + b_4y;$$

and of course the usual deductions may be made from it relative to tangent lines, as in Mr. Salmon's paper on the *Contact of Right Lines with Surfaces*, (Vol. I. p. 329 of this *Journal*).

If we equate the whole of the coefficients of (4) to zero, we obtain the conditions that the surface may contain a line passing through A and B. The meaning of these conditions can be interpreted in the mode before pointed out. Putting then $x_1x_2\ldots$, $y_1y_2\ldots$, for the roots of (4), the series of conditions is obtained by making

$$\Pi\,(xy_1 + x_1y)\ldots(xy_n + x_ny) = 0$$

independently of x and y. This results from one or more sets of roots becoming indeterminate. Now if we take three surfaces $U=0$, $V=0$, $W=0$, and eliminate first z and then ω between V and W, and substitute the values of z and ω derived from the resultant in U, we have an equation

$$\phi\,(xy) = 0$$

of the degree mnp, and equating the coefficients of this to zero we obtain the conditions that U, V, W may contain a common line. For these equations are sufficient if U contains the whole intersection of V and W, and a portion are sufficient if U, V, W contain a common line of a lower degree. We see then that the series of conditions will be given by

$$\Pi\,(xy_1 + y_1x)\ldots(xy_{nmp} + yx_{nmp}) = 0,$$

independently of x and y, a form similar to that obtained by Mr. Cayley's assumption of an arbitrary plane. An entirely arbitrary plane, however, gives superfluous conditions; for it can be subjected to two conditions and yet pass through the whole of space by rotation, for instance, round a fixed axis. Instead then of assuming a plane, we see that the required conditions are obtained by eliminating z and ω from

$$U = 0,$$
$$V = 0,$$
$$W = 0,$$

and making the resultant $=0$ independently of x and y. It is obvious the result will be of the degree

$$\mu np + \nu mp + \pi nn$$

in variables contained in the coefficients of U, V, W in degrees μ, ν, π (m, n, p being the orders of the surfaces).

The application of the foregoing notation and the development (3) to the general equation of the second order will illustrate its use. Let

$$\underset{aa}{\Delta}x^2 + \underset{bb}{\Delta}y^2 + \underset{cc}{\Delta}z^2 + \underset{dd}{\Delta}\omega^2 + 2\underset{ab}{\Delta}xy + 2\underset{ac}{\Delta}xz + 2\underset{ad}{\Delta}x\omega + 2\underset{bc}{\Delta}yz + 2\underset{cd}{\Delta}z\omega + 2\underset{bd}{\Delta}y\omega = 0 \qquad (5)$$

be the general equation, $xyz\omega$ being still supposed to contain the angular factors implicitly.

We see at once that

$$\underset{aa}{\Delta}x^2 + \underset{bb}{\Delta}y^2 + \underset{cc}{\Delta}z^2 + \underset{dd}{\Delta}\omega^2 = 0$$

implies that the tetrahedron of reference is self-conjugate. Again,

$$\underset{aa}{\Delta}x^2 + \underset{bb}{\Delta}y^2 + 2\underset{cd}{\Delta}z\omega = 0$$

implies that the plane c is the tangent plane at D, and d is the tangent plane at C, the plane a is the polar plane of A, and b the polar plane of B. It is also obvious that

$$\underset{ab}{\Delta}xy + \underset{ac}{\Delta}xz + \underset{ad}{\Delta}x\omega + \underset{bc}{\Delta}yz + \underset{bd}{\Delta}y\omega + \underset{cd}{\Delta}z\omega = 0$$

circumscribes the tetrahedron of reference.

In the surface of the second order $\underset{ab}{\Delta}$ is equal to $\underset{ba}{\Delta}$. Hence

$$\underset{ab}{\Delta}\underset{cd}{\Delta}\ldots\underset{ka}{\Delta} = \underset{ba}{\Delta}\underset{dc}{\Delta}\ldots\underset{ak}{\Delta},$$

or if p_{kl} represent the perpendicular from $k_1k_2k_3k_4$ to the polar plane of $l_1l_2l_3l_4$, we have

$$p_{ab}\cdot p_{cd}\cdots p_{ka} = p_{ba}\cdot p_{dc}\cdots p_{ak}.$$

If therefore a_1, &c., b_1, &c., k_1, &c. be points on the surface forming a gauche inscribed polygon, and we divide both members of the last given equation by the product of the sides of the polygon, it appears that if a gauche polygon be inscribed in a surface of the second degree, the continued product of the sines of the angles which each side makes with the tangent plane at its right-hand extremity is equal to the similar product of the sines of the angles made with the tangent planes at the other extremities. The corresponding theorem for conics is readily proved in an analogous way, and the equation

$$\frac{\sin\theta}{\sin\phi}\,xy + \frac{\sin\phi'}{\sin\theta'}\,xz + yz = 0,$$

the general equation of a conic circumscribing a triangle xyz, is in conformity with the theorem referred to. For multiplying by $\dfrac{\sin\phi}{\sin\theta}$, we get

$$xy + \frac{\sin\theta''}{\sin\phi''}\,xz + \frac{\sin\phi}{\sin\theta}\,yz = 0,$$

in consequence of

$$\sin\theta \sin\theta' \sin\theta'' = \sin\phi \sin\phi' \sin\phi'',$$

we have then

$$p_{ab}p_{bd}p_{dc}p_{ca} = p_{ba}p_{ab}p_{cd}p_{ac},$$

$$p_{ab}p_{bc}p_{cd}p_{da} = p_{ba}p_{cb}p_{dc}p_{ad},$$

$$p_{ad}p_{db}p_{bc}p_{ca} = p_{da}p_{bd}p_{cb}p_{ac},$$

we also have

$$\frac{\underset{ab}{\Delta}}{\underset{cb}{\Delta}} = \frac{p_{ab}}{p_{cb}},$$

and so on symmetrically.

By the aid of these considerations, we may write (5) in the following and corresponding forms

$$\frac{p_{aa}p_{ab}}{p_{ba}p_{cb}}\,x^2 + \frac{p_{bb}}{p_{cb}}\,y^2 + \frac{p_{cc}}{p_{bc}}\,z^2 + \frac{p_{dd}p_{db}}{p_{bd}p_{cb}}$$

$$+ 2\left\{\frac{p_{ab}}{p_{cb}}\,xy + \frac{p_{ac}}{p_{bc}}\,xy + \frac{p_{dc}}{p_{bc}}\,\omega z + \frac{p_{ad}}{p_{bd}}\cdot\frac{p_{db}}{p_{cb}}\,x\omega + \frac{p_{ab}}{p_{cb}}\,y\omega + yz\right\} = 0,$$

where the meaning of the coefficients very clearly appears.

Since
$$\underset{ba}{\Delta} = Rp_{ba}, \quad \underset{aa}{\Delta} = Rp_{aa},$$

$$\underset{ab}{\Delta} = R'p_{ab}, \quad \underset{bb}{\Delta} = R'p_{bb},$$

$$\underset{ab}{\Delta} = \sqrt{(RR'p_{ba}p_{ab})},$$

we obtain, precisely as in the case as conics, the equation where the edges of x, y, z, ω are tangent lines. For if the polar plane of A and the polar plane of B intersect on AB, we have geometrically

$$p_{aa}p_{bb} = p_{ab}p_{ba},$$

and the other conditions may be similarly found. The negative sign of the root gives the equation of the circumscribed surface; the positive sign makes the equation a perfect square, in which case, strictly speaking, every line is a tangent.

Let
$$x^2 + y^2 + z^2 + \omega^2 = 0 \,\ldots\ldots\ldots\ldots\ldots\ldots (a')$$
$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0 \,\ldots\ldots\ldots\ldots\ldots\ldots (b')$$

represent the same surface, constants being implicit, and let

$$Ax^2 + By^2 + Cz^2 + D\omega^2 + Exy + Fyz + Gx\omega$$
$$+ Hyz + Ky\omega + Lz\omega = 0 \ldots\ldots (c'),$$

$$A'\alpha^2 + B'\beta^2 + C'\gamma^2 + D'\delta^2 + E'\alpha\beta + F'\alpha\gamma + G'\alpha\delta$$
$$+ H'\beta\gamma + K'\beta\delta + L'\gamma\delta = 0 \ldots\ldots (d')$$

represent another surface. If

$$x = l_1\alpha + m_1\beta + n_1\gamma + p_1\delta,$$
$$y = l_2\alpha + m_2\beta + n_2\gamma + p_2\delta,$$
$$z = l_3\alpha + m_3\beta + n_3\gamma + p_3\delta,$$
$$\omega = l_4\alpha + m_4\beta + n_4\gamma + p_4\delta,$$

we shall have, by virtue of (b'),

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 = 1, \quad l_1m_1 + l_2m_2 + l_3m_3 + l_4m_4 = 0,$$
$$\&c. \qquad\qquad \&c.,$$

and therefore

$$\alpha = l_1x + l_2y + l_3z + l_4\omega,$$
$$\beta = m_1x + m_2y + m_3z + m_4\omega,$$
$$\gamma = n_1x + n_2y + n_3z + n_4\omega,$$
$$\delta = p_1x + p_2y + p_3z + p_4\omega;$$

and therefore, by virtue of (a'),

$$l_1^2 + m_1^2 + n_1^2 + p_1^2 = 1, \quad l_1l_2 + m_1m_2 + n_1n_2 = 0,$$
$$\&c. \qquad\qquad \&c.,$$

and consequently it is easily seen that

$$A + B + C + D = A' + B' + C' + D',$$

and if ∇ is the determinant of (c') and ∇' of (d'), we have, by virtue of the condition, that a plane should touch the surface

$$\frac{d\nabla}{dA}+\frac{d\nabla}{dB}+\frac{d\nabla}{dC}+\frac{d\nabla}{dD}=\frac{d\nabla'}{dA'}+\frac{d\nabla'}{dB'}+\frac{d\nabla'}{dC'}+\frac{d\nabla'}{dD'}.$$

Hence (1) if two tetrahedrons be self-conjugate with regard to a surface of the second degree, and if another surface of the second degree be circumscribed about seven of their corners, it must pass through the eighth, and (2) if a surface of the second degree be inscribed within seven of their sides, it must touch the eighth. These theorems correspond to the theorem relative to conics inscribed and circumscribed about self-conjugate triangles, and the mode of proof corresponds with that given for conics by Mr. Salmon, in his *Geometrical Notes*, contained in this *Journal*. The plane theorem may of course be proved in a similar manner. I may remark, that, as in the use of conics we may obtain the forms of (5) subject to given conditions.

With reference to the equation of a conic passing through two given points and touching a given line, a rather simpler process suggests itself than the one given by Mr. Cayley (Vol. II. p. 46) for finding the general equation of a conic passing through two given points and touching a given line, for we have immediately

$$\left|\begin{matrix} \sqrt{\begin{pmatrix} x, & y, & z \\ a, & \beta, & \gamma \\ a, & b, & c \end{pmatrix}}, & \sqrt{\begin{pmatrix} x, & y, & z \\ a, & b, & c \\ a', & \beta', & \gamma' \end{pmatrix}}, & \sqrt{(\lambda x+\mu y+\nu z)} \\ 0, & \sqrt{\begin{pmatrix} a, & \beta, & \gamma \\ a, & b, & c \\ a', & \beta', & \gamma' \end{pmatrix}}, & \sqrt{(\lambda a+\mu\beta+\nu a)} \\ \sqrt{\begin{pmatrix} a', & \beta', & \gamma' \\ a, & \beta, & \gamma \\ a, & b, & c \end{pmatrix}}, & 0, & \sqrt{(\lambda a'+\mu\beta'+\nu a')} \end{matrix}\right|=0$$

for the required equation. The imaginary quantities disappear by the alteration of the order of the constants.

By observing the meaning of the coefficients we are able to determine the form of the equation, when an angle of the tetrahedron of reference is a double point. It is obvious that if the angle xzw is such a point, the coefficients of y^n, xy^{n-1}, zy^{n-1}, wy^{n-1} vanish. The like applies to plane curves; for instance, if a curve of the fourth order has three double points at the angles of the triangle of reference, we see at once that the coefficients of x^4, y^4, z^4, xy^3, xz^3, yx^3, yz^3, zy^3, zx^3 all vanish, so that the equation takes the form

$$Ax^2y^2 + By^2z^2 + Cz^2x^2 + (Dx + Ey + Fz)\,xyz.$$

The same applies to curves of any degree, and generally, if the angles of a self-conjugate triangle be on the curve, they must be double points, as indeed is otherwise evident. Since the independent constants of transformation are six, we may put curves of the fourth order into the form

$$Ax^4 + By^4 + Cz^4 + Dxy^3 + Exz^3 + Fyz^3 + Gzy^3 + Hzx^3 + Kyx^3;$$

the relations of the triangle of reference in this case appear from the forms of the vanishing coefficients.

Stourbridge, *March*, 1857.

ON THE SIMULTANEOUS TRANSFORMATION OF TWO HOMOGENEOUS FUNCTIONS OF THE SECOND ORDER.

By A. Cayley.

In a former paper with this title, *Cambridge and Dublin Math. Journal*, t. IV. pp. 47–50, I gave (founded on the methods of Jacobi and Prof. Boole) a simple solution of the problem, but the solution may I think be presented in an improved form as follows, where as before I consider for greater convenience the case of three variables only.

Suppose that by the linear transformation*

$$(x, y, z) = \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right)(x_1, y_1, z_1),$$

* I represent in this manner the system of equations

$$x = \alpha x_1 + \beta y_1 + \gamma z_1, \text{ \&c.}$$

and so in all like cases.

we have identically

$$(a, b, c, f, g, h)(x, y, z)^2 = (a_1, b_1, c_1, f_1, g_1, h_1)(x_1, y_1, z_1)^2,$$
$$(A, B, C, F, G, H)(x, y, z)^2 = (A_1, B_1, C_1, F_1, G_1, H_1)(x_1, y_1, z_1)^2.$$

And write also

$$(\xi_1, \eta_1, \zeta_1) = \begin{vmatrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{vmatrix} (\xi\ \eta, \zeta).$$

Comparing these with the relations between (x, y, z) and (x_1, y_1, z_1), we see that

$$(\xi, \eta, \zeta)(x, y, z) = (\xi_1, \eta_1, \zeta_1)(x_1, y_1, z_1),$$

and multiplying the first of the relations between two quadrics by an indeterminate quantity λ, and adding it to the second, we have

$$(\lambda a + A, \ldots)(x, y, z)^2 = (\lambda a_1 + A_1, \ldots)(x_1, y_1, z_1)^2.$$

We have thus a linear function and a quadric transformed into functions of the same form by means of the linear substitutions, and any invariant of the system will remain unaltered to a factor près, such factor being a power of the determinant of substitution. The invariants are, 1° the discriminant of the quadric; 2° the reciprocant, considered not as a contravariant of the quadric, but as an invariant of the system. And if we write

$$\text{K} = \text{Disc.}(\lambda a + A, \ldots)(x, y, z)^2,$$

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\xi, \eta, \zeta)^2 = \text{Recip.}(\lambda a + A, \ldots)(x, y, z)^2,$$

then K_1, &c. being the analogous expressions for the transformed functions, and the determinant of substitution being represented by Π, we have

$$\text{K}_1 = \Pi^2\text{K},$$

$$(\mathfrak{A}_1, \ldots)(\xi_1, \eta_1, \zeta_1)^2 = \Pi^2(\mathfrak{A}, \ldots)(\xi, \eta, \zeta)^2,$$

and substituting for ξ_1, η_1, ζ_1 their values in terms of ξ, η, ζ, the last equation breaks up into six equations, and we have

$$\text{K}_1 = \Pi^2\text{K},$$

$$(\mathfrak{A}_1, \ldots)(\alpha, \alpha', \alpha'')^2 = \Pi^2\mathfrak{A},$$

$$\vdots$$

$$(\mathfrak{A}_1, \ldots)(\beta, \beta', \beta'')(\gamma, \gamma', \gamma'') = \Pi^2\mathfrak{F},$$

$$\vdots$$

which is the system obtained in a somewhat different manner in my former paper. Putting $f_1 = g_1 = h_1 = F_1 = G_1 = H_1 = 0$, and writing also (which is no additional loss of generality) $a_1 = b_1 = c_1 = 1$, the formulæ become

$$(a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2 = (1,\ 1,\ 1)(x_1^2,\ y_1^2,\ z_1^2),$$

$$(A,\ B,\ C,\ F,\ G,\ H)(x,\ y,\ z)^2 = (A_1,\ B_1,\ C_1)(x_1^2,\ y_1^2,\ z_1^2),$$

viz. there are two given quadrics which are to be by the same linear substitution transformed, one of them into the form $x_1^2 + y_1^2 + z_1^2$ and the other into the form $A_1x_1^2 + B_1y_1^2 + C_1z_1^2$, where A_1, B_1, C_1 have to be determined. The solution is contained in the following system of formulæ, viz.

$$(A_1 + \lambda)(B_1 + \lambda)(C_1 + \lambda) = \Pi^2 \text{ Disc. } (\lambda a + A, \ldots),$$

which gives A_1, B_1, C_1 as the roots of a cubic equation, and gives also

$$1 = \Pi^2 \text{ Disc. } (a, \ldots) = \Pi^2\kappa \text{ or } \Pi^2 = \frac{1}{\kappa} \text{ suppose,}$$

and we have then, writing for shortness, $(\bullet)(X,\ Y,\ Z)$ for

$$\{(B_1+\lambda)(C_1+\lambda),\ (C_1+\lambda)(A_1+\lambda),\ (A_1+\lambda)(B_1+\lambda)\}(X,\ Y,\ Z),$$

$$(\bullet)(\alpha^2,\quad \alpha'^2,\quad \alpha''^2) = \frac{1}{\kappa}\,\mathfrak{A},$$

$$(\bullet)(\beta^2,\quad \beta'^2,\quad \beta''^2) = \frac{1}{\kappa}\,\mathfrak{B},$$

$$(\bullet)(\gamma^2,\quad \gamma'^2,\quad \gamma''^2) = \frac{1}{\kappa}\,\mathfrak{C},$$

$$(\bullet)(\beta\gamma,\ \beta'\gamma',\ \beta''\gamma'') = \frac{1}{\kappa}\,\mathfrak{F},$$

$$(\bullet)(\gamma\alpha,\ \gamma'\alpha',\ \gamma''\alpha'') = \frac{1}{\kappa}\,\mathfrak{G},$$

$$(\bullet)(\alpha\beta,\ \alpha'\beta',\ \alpha''\beta'') = \frac{1}{\kappa}\,\mathfrak{H},$$

where $(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H})$ are the coefficients of the reciprocant of $(\lambda a + A, \ldots)(x,\ y,\ z)^2$. Writing $\lambda = -A_1$, $-B_1$, or $-C_1$ the quadric functions on the left-hand side become mere monomials, and we have the actual values of the squares and products α^2, $\beta\gamma$, &c. of the coefficients of the linear substitutions: thus α^2, β^2, γ^2, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ are respectively equal to $\mathfrak{A}_0$, $\mathfrak{B}_0$, $\mathfrak{C}_0$, $\mathfrak{F}_0$, $\mathfrak{G}_0$, $\mathfrak{H}_0$ each into the common factor $\frac{1}{\kappa}(B_1 - A_1)(C_1 - A_1)$, the suffix denoting that we are to write in the expressions for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ the value $-A_1$ for λ; and similarly for the sets $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$.

2, Stone Buildings,
27 *March*, 1857.

A WORD ON FOCI.

By Samuel Roberts, M.A.

1. If the n real foci of a curve of the n^{th} class be taken two and two, to each pair of real foci so taken corresponds a pair of imaginary foci, lying on a real line bisecting and perpendicular to the line joining the two real foci and at equal imaginary distances on either side thereof, as is known in the case of conics. The general theorem follows from considering the points of intersection, and the lines joining them, of

$$(y - y_1) \pm \sqrt{(-1)}\,(x - x_1) = 0,$$

$$(y - y_2) \pm \sqrt{(-1)}\,(x - x_2) = 0,$$

or the evanescent circles

$$(y - y_1)^2 + (x - x_1)^2 = 0,$$

$$(y - y_2)^2 + (x - x_2)^2 = 0.$$

Hence, if two real foci coincide, the two corresponding imaginary foci coincide with the resulting real double focus, and $2(n-2)$ single imaginary foci coincide in $2(n-2)$ double imaginary foci, and the total number of foci is $(n-1)^2$.

Generally, if m real foci coincide, then the corresponding $m(m-1)$ imaginary foci coincide in the same point, and $2(m-1)(n-m)$ single imaginary foci coincide with $2(n-m)$ imaginary foci, forming $2(n-m)$ imaginary foci of the m^{th} degree.

2. "All curves of the third degree whose highest terms are divisible by x^2+y^2; and all curves of the fourth degree whose equations are of the form

$$(x^2+y^2)^2+u_1(x_2+y_2)+u_2=0$$

have four foci lying on a circle," a known theorem.

3. If four real foci lie on a circle, then the imaginary foci corresponding to the real foci taken in pairs lie four and four on circles cutting the circle whereon the four real foci lie at right angles.

For let $ABCD$ the real foci lie on the circle $ABCD$ with centre R, and let AB, CD produced meet in O. Draw radii $RP.RQ$ perpendicular respectively to $AB.CD$, and let

$$PB=c, \quad QD=c', \quad OP=t, \quad OQ=t'.$$

Then if r be the radius of circle with centre O and cutting circle $ABCD$ at right angles, we have

$$r^2=t^2-c^2=t'^2=c'^2=t^2+\{\sqrt{(-1)}\,c\}^2=t'^2+\{\sqrt{(-1)}\,c'\}^2.$$

But the imaginary foci corresponding to A, B lie on RP at distances $\pm\sqrt{(-1)}\,c$ from P, and the imaginary foci corresponding to C, D, lie on RQ at distances $\pm\sqrt{(-1)}\,c'$ from Q. Therefore the circle whose centre is O and radius r, passes through the imaginary foci corresponding to A, B, and C, D.

Wherefore, applying similar reasoning to the pairs AD, BC, BD, AC, the above theorem holds good.

4. The circles whereon the corresponding imaginary foci lie cut one another at right angles mutually.

For let $$x^2+y^2=a^2$$

be the equation of circle $ABCD$, and

$$(x-x_1)^2+(y-y_1)^2=x_1^2+y_1^2-a^2 \ldots\ldots\ldots\ldots(1),$$
$$(x-x_2)^2+(y-y_2)^2=x_2^2+y_2^2-a^2 \ldots\ldots\ldots\ldots(2)$$

be the equations of circles on which two corresponding set of four imaginary foci lie.

Transform to centre of circle (1), whereby

(1) becomes $x^2+y^2=x_1^2+y_1^2-a^2$,

(2) becomes

$$(x+x_1-x_2)^2+(y+y_1-y_2)^2=x_2^2+y_2^2-a^2$$
$$=(x_1-x_2)^2+(y_1-y_2)^2-x_1^2-y_1^2+a^2,$$

if $$x_1x_2+y_1y_2=a^2.$$

But this is so because x_1y_1 is on the polar of x_2y_2 with respect to $x^2+y^2=a^2$, and therefore (1) and (2) cut one another at right angles. Similar reasoning applies to the remaining circles, whence the above theorem.

And considering the nature of circles so related we are able to conclude amongst other inferences, that the twelve imaginary foci lie on six right lines passing through the centre of the circle whereon the real foci lie, and therefore intersecting on the curve (Salmon), and generally, the circles being symmetrically related, that twelve corresponding foci lie in lines passing through the centre of the circle whereon the remaining four lie.

Also the triangle formed by the centres of any three of the circles is self-conjugate with respect to the fourth, and one of the circles must be imaginary; namely, that the centre of which is the intersection of the diagonals of the quadrilateral *ABCD*, moreover the radical axes of any three pass through the centre of the fourth.

However, the properties of four circles cutting one another mutually at right angles, though worth observing, are independent of the nature of the curves whose foci lie thereon, and therefore more proper subjects of separate investigation. It will be observed that the before-mentioned properties of imaginary foci are due to the disposition of the real foci, their origin and only indirectly dependent on the curve.

5. If the four real foci of a bicircular quadratic lie on the same right line, the double foci lie also thereon.

For take the line as axis of x and let the equation of the curve be of the form

$$l\sqrt{\{(x-a)^2+y^2\}}+m\sqrt{\{(x-l)^2+y^2\}}+n\sqrt{\{(x-c)^2+y^2\}}=0,$$

or

$$l\sqrt{(A)}+m\sqrt{(B)}+n\sqrt{(C)}=0,$$

equivalent to

$$l''A^2+m''B^2+n''C^2-2l^2m^2AB-2l^2n^2AC-2m^2n^2BC=0,$$

this may be put into the form

$$(x^2+y^2)^2+(x^2+y^2)(ex+f)+gx^2+hx+k=0,$$

and further into the form

$$\{(x^2+y^2+ex+f)+\sqrt{(g-e)}\,x\}\{(x^2+y^2+ex+f)-\sqrt{(g-e)}\,x\}$$
$$+(h-2f)\,x+k-f^2=0,$$

wherefrom we gather that the coordinates of the centres of the circles are

$$y=0, \quad x=-\frac{e+\sqrt{(g-e)}}{2}, \quad y=0, \quad x=-\frac{e-\sqrt{(g-e)}}{2},$$

and these being the double foci the proposition is true.

Ovals of Descartes and Cassini are of this class.

It may be remarked that the triple focus of a Cartesian oval is that point whereof the polar cosine is a circle.

Nov. 1857.

NOTE ON A FORMULA IN FINITE DIFFERENCES.

By A. Cayley.

In Jacobi's Memoir '*De usu Legitimo Formulæ Summatoriæ Maclaurinianæ*,' Crelle, t. XII. pp. 263–273 (1834), expressions are given for the sums of the odd powers of the natural numbers 1, 2, 3...x in terms of the quantity

$$u=x(x+1),$$

viz. putting for shortness

$$Sx^r = 1^r + 2^r + \ldots + x^r,$$

the expressions in question are

$$Sx^3 = \tfrac{1}{4}u^2,$$
$$Sx^5 = \tfrac{1}{6}u^2\left(u-\tfrac{1}{2}\right),$$
$$Sx^7 = \tfrac{1}{8}u^2\left(u^2-\tfrac{4}{3}u+\tfrac{2}{3}\right),$$
$$Sx^9 = \tfrac{1}{10}u^2\left(u^3-\tfrac{5}{2}u^2+3u-\tfrac{3}{2}\right),$$
$$Sx^{11} = \tfrac{1}{12}u^2\left(u^4-4u^3+\tfrac{17}{2}u^2-10u+5\right),$$
$$Sx^{13} = \tfrac{1}{14}u^2\left(u^5-\tfrac{35}{6}u^4+\tfrac{287}{15}u^3-\tfrac{118}{3}u^2+\tfrac{691}{15}u-\tfrac{691}{30}\right),$$

&c.,

which, especially as regards the lower powers, are more simple than the ordinary expressions in terms of x.

The expressions are continued by means of a recurring formula, viz. if

$$Sx^{2p-3} = \frac{1}{2p-2}\left\{u^{p-1} - a_1 u^{p-2} \ldots + (-)^{p-1} a_{p-3} u^2\right\},$$

$$Sx^{2p-1} = \frac{1}{2p}\left\{u^{p} - b_1 u^{p-2} \ldots + (-)^{p} b_{p-2} u^2\right\},$$

then

$$\begin{aligned}
2p(2p-1)a_1 &= (2p-2)(2p-3)b_1 - p(p-1),\\
2p(2p-1)a_2 &= (2p-4)(2p-5)b_2 - (p-1)(p-2)b_1,\\
2p(2p-1)a_3 &= (2p-6)(2p-7)b_3 - (p-2)(p-3)b_2,\\
&\vdots\\
2p(2p-1)a_{p-3} &= 5.6\ b_{p-3} - 3.4\ b_{p-4},\\
0 &= 3.4\ b_{p-2} - 2.3\ b_{p-3},
\end{aligned}$$

by means of which the coefficients b can be determined when the coefficients a are known.

Jacobi remarks also that the expressions for the sums of the even powers may be obtained from those for the odd powers by means of the formula

$$Sx^{2p} = \frac{1}{2p+1} d_x Sx^{2p+1},$$

which shews that any such sum will be of the form $(2x+1)u$ into a rational and integral function of u: thus in particular

$$Sx^2 = \tfrac{1}{6}(2x+1)u.$$

To shew *à priori* that Sx^{2p+1} can be expressed as a rational and integral function of u, it may be remarked that $Sx^{2p+1} = \phi_1 x$ where $\phi_1 x$ denotes the summatory integral $\Sigma(x+1)^{2p+1}$, taken so as to vanish for $x=0$: $\phi_1 x$ is a rational and integral function of x of the degree $2p+2$, and which, as is well known, contains x^2 as a factor. Suppose that y is any positive or negative integer less than x, we have

$$\phi_1 x - \phi_1 y = (y+1)^{2p+1} + (y+2)^{2p+1} \ldots + x^{2p+1},$$

and in particular putting $y = -1 - x$,

$$\phi_1 x - \phi_1(-1-x) = (-x)^{2p+1} + (1-x)^{2p+1} \ldots + x^{2p+1} = 0,$$

since the terms destroy each other in pairs; we have therefore $\phi_1 x = \phi_1(-1-x)$. Now $u = x^2 + x$, or writing this equation under the form $x^2 = -x + u$, we see that any rational and integral function of x may be reduced to the form $Px + Q$, where P and Q are rational and integral functions of u. Write therefore $\phi_1 x = Px + Q$: the substitution of $-1-x$ in the place of x leaves u unaltered, and the equation $\phi_1 x = \phi_1(-1-x)$ thus shews that $P=0$; we have therefore $\phi_1 x = Q$ a rational and integral function of u. Moreover $\phi_1 x$ as containing the factor x^2, must clearly contain the factor u^2, and the expressions for Sx^{2p+1} are thus shown to be of the form given by Jacobi.

We may obtain a finite expression for Sx^n in terms of the differences of 0^n as follows: we have

$$Sx^n = 1^n + 2^n \ldots + x^n = \{(1+\Delta) + (1+\Delta)^2 \ldots + (1+\Delta)^x\} 0^n$$

$$= \frac{1+\Delta}{\Delta}\{(1+\Delta)^x - 1\} 0^n,$$

and putting $(1+\Delta)^x = e^{x \log(1+\Delta)}$ and observing that the term independent of x vanishes, and that the terms containing powers higher than x^{n+1} also vanish, we have

$$Sx^n = S_k \left\{\frac{1+\Delta}{\Delta} \log^k (1+\Delta)\right\} 0^n . \frac{x^k}{\Pi k},$$

where the summation with respect to k, extends from $k=1$ to $k=n+1$, or what is the same thing (since the term corresponding to $k=1$ in fact vanishes) from $k=2$ to $k=n+1$.

The equation $x^2 = -x + u$ gives

$$x^k = P_k x + Q_k,$$

and it is easy to see that writing for shortness

$$M_k = 1 + \frac{k-3}{1} u + \frac{k-4 . k-5}{1.2} u^2 + \frac{k-5 . k-6 . k-7}{1.2.3} u^3 + \ldots,$$

where the series is to be continued to the term $u^{\frac{1}{2}(k-2)}$ or $u^{\frac{1}{2}(k-3)}$ according as k is even or odd, we have

$$P_k = (-)^{k+1} M_{k+1}, \quad Q_k = (-)^k u M_k,$$

we have consequently

$$Sx^n = x S_k \left\{\frac{1+\Delta}{\Delta} \log^k (1+\Delta)\right\} 0^n . \frac{(-)^{k+1} M_{k+1}}{\Pi k}$$

$$+ S_k \left\{\frac{1+\Delta}{\Delta} \log^k (1+\Delta)\right\} 0^n . \frac{(-)^k u M_k}{\Pi k}.$$

If n is odd, $=2p+1$, then (by what precedes) the first term vanishes, or we have

$$S_k \left\{\frac{1+\Delta}{\Delta} \log^k (1+\Delta)\right\} 0^{2p+1} \frac{(-)^{k+1} M_{k+1}}{\Pi k} = 0, \ (k=1 \text{ to } k=2p+2),$$

and the formula becomes

$$Sx^{2p+1} = S_k \left\{\frac{1+\Delta}{\Delta} \log^k (1+\Delta)\right\} 0^{2p+1} \frac{(-)^k u M_k}{\Pi k}, \qquad (k=1 \text{ to } k=2p+2),$$

which it may be noticed puts in evidence the factor u but not the factor u^2.

If n is even, $= 2p$, then (by what precedes) the coefficient of x is to the constant term in the ratio 2 : 1, or we have

$$S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p}\frac{(-)^{k+1}(M_{k+1}-2uM_k)}{\Pi k}=0,\ (k=1 \text{ to } k=2p+1),$$

and the formula becomes

$$Sx^{2p}=(2x+1)S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p}\frac{(-)^k uM_k}{\Pi k},\ (k=1 \text{ to } k=2p+1).$$

The values of the functions M are as follows:

$$M_1 = 0,$$
$$M_2 = 1,$$
$$M_3 = 1,$$
$$M_4 = 1 + u,$$
$$M_5 = 1 + 2u,$$
$$M_6 = 1 + 3u + u^2,$$
$$M_7 = 1 + 4u + 3u^2,$$
&c.

As a simple example of the formulæ, we have

$$Sx^3 = \left\{\frac{1+\Delta}{\Delta}\log^2(1+\Delta)\right\}0^3.\tfrac{1}{2}u$$
$$+\left\{\frac{1+\Delta}{\Delta}\log^3(1+\Delta)\right\}0^3.-\tfrac{1}{6}u$$
$$+\left\{\frac{1+\Delta}{\Delta}\log^4(1+\Delta)\right\}0^3.\tfrac{1}{24}(u+u^2),$$

and the coefficients are

$$(\Delta - \tfrac{1}{12}\Delta^3)\,0^3 = 1 - \tfrac{1}{12}6 = \tfrac{1}{2},$$
$$(\Delta^2 - \tfrac{1}{2}\Delta^3)\,0^3 = 6 - \tfrac{1}{2}6 = 3,$$
$$\Delta^3\ 0^3 = 6,$$

and therefore

$$Sx^3 = \tfrac{1}{4}u - \tfrac{1}{2}u + \tfrac{1}{4}(u+u^2) = \tfrac{1}{4}u^2,$$

which is right; the example shews however that the calculation for the higher powers would be effected more readily by means of Jacobi's recurring formula.

2, Stone Buildings, *27th Oct.*, 1857.

ON THE INCOMMENSURABILITY OF THE PERIMETER AND AREA OF SOME REGULAR POLYGONS TO THE RADIUS OF THE INSCRIBED OR CIRCUMSCRIBED CIRCLE.

By M. E. PROUHET, Professor of Mathematics at Paris.

THE present paper has for its object the exposition of an extension of some theorems given by M. Terquem, in Liouville's "*Journal of Pure and Applied Mathematics*," (First Series, t. III. p. 477.), and may be considered as a sequel to a note of mine inserted in the same Journal, *On the Arcs of Circles which have their tangents expressed by rational numbers* (Second Series, t. I. p. 215).

THEOREM I.

If m be prime to n, $\tan\frac{m\pi}{n}$ *is a root of an irreducible equation of the* $(n-1)^{\text{th}}$ *degree at most, having all its roots real.*

Demonstration. Let

$$\tan\frac{m\pi}{n}=x;$$

then $\tan n\left(\frac{m\pi}{n}\right)=0$

$$=\frac{nx-\frac{n(n-1)(n-2)}{1.2.3}x^3+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}x^5-\text{etc.}}{1-\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^4-\text{etc.}}.$$

Consequently $\tan\frac{m\pi}{n}$ or x is a root of the equation

$$(1)\quad n-\frac{n(n-1)(n-2)}{1.2.3}x^2+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}x^4-\text{etc.}=0,$$

of which the last term is x^{n-2} or nx^{n-1} according as n is an even or odd number.

Now it is obvious that the equation (1) has for its roots the real values

$$\tan\frac{\pi}{n},\quad \tan\frac{2\pi}{n},\quad \ldots\ldots \tan\frac{(n-2)\pi}{n},\quad \tan\frac{(n-1)\pi}{n},$$

$$\left(\tan\frac{\frac{n}{2}\pi}{n} = \tan\frac{\pi}{2} = \infty,\ \text{being omitted if } n \text{ be even}\right).$$

Therefore, if the equation (1) be an irreducible one, the theorem is demonstrated; if not, that equation will be the product of two or more irreducible equations, of degrees inferior to the $(n-1)^{\text{th}}$, but having all their roots real, and among them will be found the root $\frac{m\pi}{n}$.

THEOREM II.

If $\frac{m}{n}$ *be an irreducible fraction and n any number,* 1, 2, 3 *or* 6 *excepted,* $\tan^k\frac{m\pi}{n}$ *will be expressed by an irrational number.*

In the first place, let n be a prime number greater than 3. Then $\tan\frac{m\pi}{n}$ will be a root of the equation

$$(2)\quad x^{n-1} - \frac{n(n-1)(n-2)}{1.2.3}x^{n-3} + \ldots\ldots \pm \frac{n(n-1)(n-2)}{1.2.3}x^2 \mp n = 0,$$

(that is, the equation (1) reversed). But M. Eisenstein has demonstrated the following theorem: "If the coefficient of the leading term of an equation be unity, and all the other coefficients whole numbers that can be divided by the prime number n; if the last term cannot be divided by n^2, the equation is an irreducible one." All these conditions are fullfilled by the equation (2) which is therefore irreducible.

Hence it follows that $\tan^k\frac{m\pi}{n}$ is irrational, that is, $\tan\frac{m\pi}{n}$ cannot satisfy the binomial equation

$$x^k - R = 0,$$

R being a rational number, for the binomial $x^k - R$ cannot obviously be divided, if $n - 1 > 2$, by the left-hand side of an equation of the $(n-1)^{\text{th}}$ degree, *having all its roots real.*

P 2

In the second place, let $n = hp$, p being a prime number > 3. We have

$$\tan^2\frac{m\pi}{p} = \tan^2 h\left(\frac{m\pi}{hp}\right) = \frac{\tan^2\frac{m\pi}{hp}\left\{p - \frac{p(p-1)(p-2)}{1.2.3}\tan^2\frac{m\pi}{hp} + \ldots\right\}^2}{\left\{1 - \frac{p(p-1)}{1.2}\tan^2\frac{m\pi}{p} + \ldots\right\}^2},$$

from which we may conclude that $\tan^2\frac{m\pi}{hp}$ is irrational: for, if it could be rational, $\tan^2\frac{m\pi}{p}$ would be rational, which is impossible by the first part of the demonstration. Therefore the irreducible equation

$$\phi(x) = 0,$$

which $\tan\frac{m\pi}{hp}$ must satisfy (Theorem I), is a *complete* equation of the second degree, and then $\phi(x)$ cannot divide the binomial $x^k - R$. Consequently $\tan^k\frac{m\pi}{hp}$ cannot be rational.

In the third place, from $\tan^2\frac{m\pi}{8}$, $\tan^2\frac{m\pi}{9}$, $\tan^2\frac{m\pi}{12}$ being irrational, it may be shown, in the same manner, that $\tan^k\frac{m\pi}{4h}$, $\tan^k\frac{m\pi}{3h}$, $\tan^k\frac{m\pi}{6h}$ are irrational, if h be > 1, and the second theorem is entirely demonstrated.

Theorem III.

Among regular polygons, circumscribing a circle of which the radius $= 1$,

1°. *Of the square only, the perimeter and the area are rational.*

2°. *Of the triangle, square, and hexagon, only, the second powers of the perimeter and area are rational.*

3°. *Of all the other regular polygons, neither the perimeter nor the area are rational.*

Demonstration. p being the perimeter and s the area of a regular polygon of n sides, circumscribing a circle of which the radius $= 1$, we have

$$p = 2n\tan\frac{\pi}{n}, \quad s = n\tan\frac{\pi}{n}$$

irrational expressions in the above-mentioned cases, from the Theorem II.

THEOREM IV.

If $\tan^k \frac{m\pi}{n}$ *is irrational,* $\sin^k \frac{m\pi}{n}$ *is irrational.*

We have identically

$$\sin^k \frac{m\pi}{n} = \frac{\tan^k \frac{m\pi}{n}}{\left(1 + \tan^2 \frac{m\pi}{n}\right)^{\frac{1}{2}k}}.$$

If the left-hand side were any rational number R, $\tan \frac{m\pi}{n}$ would be a root of the equation

$$R^2 (1 + x^2)^k = x^{2k},$$

or

$$R^2 \left(\frac{1}{x^2} + 1\right)^k = 1.$$

That equation must have only two real roots, equal in absolute value, but with opposite signs. Hence the irreducible equation which $\tan \frac{m\pi}{n}$ must satisfy, would be a binomial equation, but this is impossible, for by hypothesis, no power of $\tan \frac{m\pi}{n}$ is rational. Therefore, etc.

THEOREM V.

Among the regular polygons inscribed in a circle of which the radius $= 1$,

1°. *Of the hexagon only, the perimeter is rational.*

2°. *Of the triangle, square, and hexagon, only, the second power of the perimeter is rational.*

3°. *Of the square, and dodecagon, only, the area is rational.*

4°. *Of the square, triangle, hexagon, octagon, and dodecagon, only, the second power of the area is rational.*

5°. *The foregoing polygons excepted, no power of the perimeter or the area is rational.*

Demonstration. This follows from the trigonometrical expressions

$$\text{perimeter} = 2n \sin \frac{\pi}{n},$$

$$\text{area} = \tfrac{1}{2}n \sin \frac{2\pi}{n},$$

and from the Theorems II. and IV.

General Scholium.

I have made many attempts to deduce the incommensurability of π, from the foregoing principles; but unfortunately, the circumstance that $n \sin \frac{\pi}{n}$, $n \tan \frac{\pi}{n}$ are always irrational, is not sufficient to shew that the limits of these expressions for $n = \infty$, are also irrational: for many examples can be given of expressions always irrational of which the limits are rational.

$$\left\{\text{Ex. } \sqrt{\left(4 + \frac{1}{n^2}\right)}, \text{ when } n = \infty\right\}.$$

Therefore some other principles are wanted in the question.

The only demonstration of the incommensurability of π that I am acquainted with, is a beautiful one, given by Lambert and a little modified by Legendre. The attempt of T. R. Young in his Mathematical Dissertations (p. 117) I do not regard as clearly satisfactory, because it is grounded on the identity of two transcendental equations which have, in truth, an infinite number of common roots, but may have another infinite number of roots not common to the two equations. On this account, the method of proof referred to appears deficient in logical accuracy.

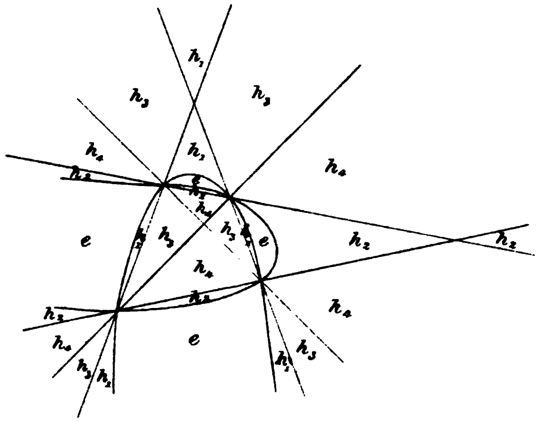
ON THE SYSTEM OF CONICS WHICH PASS THROUGH THE SAME FOUR POINTS.

By A. Cayley.

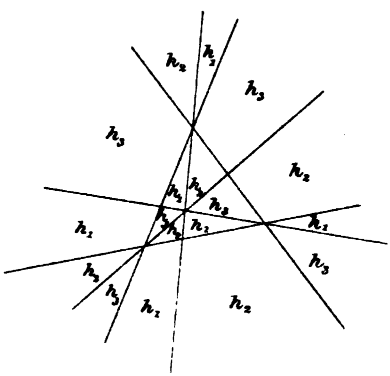
I consider the system of conics passing through the same four points; these points may be real or imaginary, but it is assumed that there is a real system of conics, this will in fact be the case if two conics of the system are real. The four points are therefore given as the points of intersection of two real conics, and it will be proper to assume in the first instance that the conics intersect in four separate and distinct points, none of them at infinity. The four points may be all real, or two real and two imaginary, or all imaginary.

First, if the points are all real, we have here two cases, viz. each of the points may lie outside of the triangle formed by the other three, or as this may be expressed, the points

A.

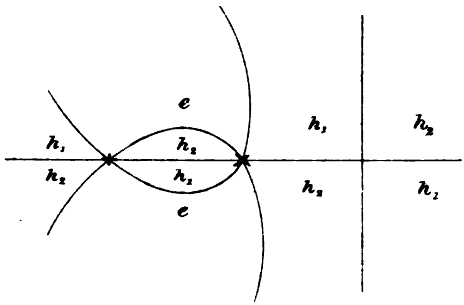


A′.

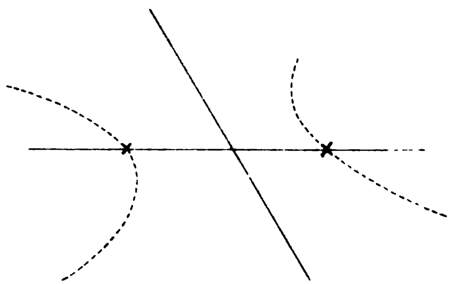


W Metcalfe Litho.

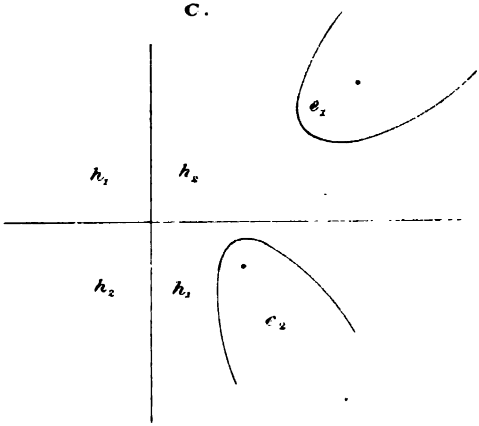
B.



B′.



C.



may form a convex quadrangle; or else one of the points may be inside the triangle formed by the other three, or as this may be expressed, the points may form a triangle and interior point. In each case the pairs of lines joining the points, two and two together, will be conics (degenerate hyperbolas) forming part of the system of conics. Consider the two cases separately.

Fig. A. Four real points forming a convex quadrangle. The system contains two parabolas, and the pairs of lines and the parabolas divide the plane of the figure into five distinct regions, one of which contains only ellipses, and the other four contains each of them hyperbolas.

Fig. A'. Four real points forming a triangle and interior point. The system does not contain any parabolas, the three pairs of lines divide the plane of the figure into three distinct regions, each of which contains only hyperbolas.

Next, if the points are two of them real and two of them imaginary. The line joining the two imaginary points will be real and this line may meet the line joining the two real points, in a point outside the two real points, or included between them, i.e. the real centre of the quadrangle may lie outside the real points, or may be included between them. Consider the two cases separately.

Fig. B. Two real and two imaginary points, the real centre of the quadrangle lying outside the real points. The system contains two parabolas, and these with the line joining the two real points and the line joining the two imaginary points divide the plane of the figure into three regions, one of which contains ellipses and the other two contains each of them hyperbolas.

Fig. B'. Two real and two imaginary points, the real centre of the quadrangle lying between the real points. There are no parabolas, and the system contains only hyperbolas.

Lastly, when the four points are imaginary. We have here only a single case.

Fig. C. Four imaginary points. The points lie on two real lines, there are (besides the point of intersection of these lines) two other real centres of the quadrangle, which lie harmonically with respect to the two lines. The system contains two parabolas and these and the two lines divide the plane of the figure into four regions, two of which contain each of them ellipses, and the other two contain each of them hyperbolas.

THEOREMS RESPECTING THE POLAR CONICS OF CURVES OF THE THIRD DEGREE.

By the Rev. T. St. Lawrence Smith, B.A.

In '*A Treatise on the Higher Plane Curves*' the author, the Rev. G. Salmon, has given a method of forming polar curves, from the first up to the $(n-1)^{\text{th}}$ degree, for any point with reference to a fixed curve of the n^{th} degree; the large number of subjects which this work embraced, prevented the author from giving any more than a very general description of these polar curves; but it has occurred to me that some interesting results might perhaps be obtained from their investigation, in the particular case when the curve with respect to which they have been formed, is but of the third degree.

I shall call the distances of any point from the curve, measured along a transversal,

$$\rho_1,\ \rho_2,\ \rho_3,$$

and the distances of the same point from its polar conic, measured along the same transversal,

$$r_1,\ r_2;$$

the relations between these lengths are given by the equation

$$3\frac{1}{r^2} - 2\frac{1}{r}\Sigma\frac{1}{\rho} + \Sigma\frac{1}{\rho_1\rho_2} = 0,$$

which is equivalent to the two following:

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{3}\left\{\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3}\right\} \text{...............(I)},$$

$$\frac{1}{r_1 r_2} = \frac{1}{3}\left\{\frac{1}{\rho_1\rho_2} + \frac{1}{\rho_2\rho_3} + \frac{1}{\rho_3\rho_1}\right\} \text{.......(II)}.$$

To find geometrically in what cases the polar conic is an hyperbola, suppose the transversal to have been drawn parallel to an asymptote, in this case one of the two intercepts r_2 becomes infinite, and (II) becomes

$$\frac{1}{\rho_1\rho_2} + \frac{1}{\rho_2\rho_3} + \frac{1}{\rho_3\rho_1} = 0,$$

or

$$\frac{\rho_1 + \rho_2 + \rho_3}{\rho_1\rho_2\rho_3} = 0 \text{..............(III)};$$

therefore, either

$$\rho_1 + \rho_2 + \rho_3 = 0,$$

or some *two* of the radii vectores ρ_1, ρ_2, ρ_3 are infinite, that is, *any point, whose polar conic is an hyperbola, lies either on some diameter or on an asymptote to the curve.* The fact of a diameter being the polar line of the point at infinity in the conjugate direction affords another proof of this; for, if a point lie on a polar conic, its polar line passes through a fixed point, the pole of the conic; if then a point at infinity lies on a polar conic, its polar line, that is, the diameter conjugate to its direction passes through the pole of the conic, so that as before the polar conic will pass through a real point at infinity if its pole lies on a real diameter.

The polar line of any point, lying on a fixed line, touches a fixed conic (see '*Higher Plane Curves,*' p. 151), hence every diameter, as being the polar line of a point at infinity, touches a fixed conic; it follows therefore that the polar conic of any point will be an hyperbola when two real tangents can be drawn from this point to the fixed conic; if the point lie on the fixed conic itself, the polar conic will be a parabola; when the tangents are imaginary it will be an ellipse.

Hence, to find the locus of all points, whose polar conics are hyperbolas having one of their asymptotes parallel to a fixed direction, find the diameter conjugate to it, and it will be the locus required.

The intersection of two diameters conjugate to fixed directions, will evidently be a point whose polar conic is an hyperbola, both of whose asymptotes are fixed in direction.

The polar conic of any point lying on an asymptote will always have a real point at infinity, if the curve have a double point at infinity; a line joining any point to this double point will be in this sense an asymptote to the curve, that is, two of its intercepts will be infinite; hence, with respect to such a curve, no point can have a polar ellipse. This follows too from the consideration that in this case the polar conic must pass through the double point, *i.e.* must have a real point at infinity.

If the curve have a cusp at infinity, since the first polar of every point must pass through the cusp, and have its tangent the same as the tangent at the cusp, it appears that one asymptote of every polar conic is fixed; in this case the polar conic can never be a parabola, unless the tangent be altogether at infinity, when it will always be a parabola.

We shall show farther on, that such a curve is the only one of the third degree, with respect to which the polar conic of any point is a parabola.

Equation (III) is evidently satisfied by

$$\rho_1 = \infty, \quad \rho_2 = \infty, \quad \rho_3 = \infty,$$

in this case (I) becomes

$$\frac{1}{r_1} + \frac{1}{r_2} = 0,$$

or as $\frac{1}{r_2}$ has been already assumed equal to zero

$$\frac{1}{r_1} = 0;$$

therefore the polar conic of any point lying on a tangent to a double point, cusp, or point of inflexion, at infinity, is an hyperbola one of whose asymptotes is the tangent on which the point lies.

To find in general when this shall be the case, namely, that the polar conic of a point shall be an hyperbola on one of whose asymptotes it shall lie, since

$$\frac{1}{r_1} = 0, \quad \frac{1}{r_2} = 0,$$

we have simultaneously, from (I) and (II),

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = 0,$$

$$\frac{1}{\rho_1\rho_2} + \frac{1}{\rho_2\rho_3} + \frac{1}{\rho_3\rho_1} = 0,$$

or

$$\frac{\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1}{\rho_1\rho_2\rho_3} = 0,$$

$$\frac{\rho_1 + \rho_2 + \rho_3}{\rho_1\rho_2\rho_3} = 0,$$

either therefore

$$\rho_1 = \infty, \quad \rho_2 = \infty, \quad \rho_3 = \infty,$$

the case we have just discussed, or

$$\left.\begin{array}{r} \rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 = 0 \\ \rho_1 + \rho_2 + \rho_3 = 0 \end{array}\right\} \ldots\ldots\ldots\ldots \text{(IV)},$$

that is, the point is an intersection of a diameter and a diametral conic, both conjugate to the direction of the asymptote.

Hence, to find a point such that its polar conic shall be an hyperbola, one of whose asymptotes passes through the point in a fixed direction, construct the diameter and diametral conic respectively conjugate to this direction, either of their intersections (if real) will be the point required.

This asymptote will meet the curve in one real and two imaginary points, for the result of eliminating ρ_3 between the two equations (IV) is

$$\rho_1^2 + \rho_1\rho_2 + \rho_2^2 = 0,$$

an equation whose roots are imaginary; hence no such point can lie within an oval or loop (if the curve have such), or in any place where a line through it must meet the curve in three real points.

Since through no point but the center of a conic, can there be drawn two chords to be bisected at the point, it follows that any point will be the center of its own polar conic, when through the point there can be drawn two transversals for which r_1 and r_2 have equal and opposite values, and also that if two such transversals can be drawn, every transversal through the point will give this relation between r_1 and r_2; for any such transversal we have

$$r_1 + r_2 = 0;$$

therefore $$\frac{1}{r_1} + \frac{1}{r_2} = 0;$$

and therefore from (I)

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = 0,$$

or $$\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 = 0,$$

since the other solution

$$\rho_1 = \infty, \quad \rho_2 = \infty, \quad \rho_3 = \infty,$$

(the case just discussed) will not satisfy

$$r_1 + r_2 = 0,$$

though it does satisfy

$$\frac{1}{r_1} + \frac{1}{r_2} = 0.$$

Since the equation

$$\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1 = 0$$

must be satisfied for two directions, such a point as we are seeking must be an intersection of two diametral conics, and since when true for two directions, it must be true for all, it follows that all the diametral conics must intersect in the same points; this is true, for they all are the polar conics of points on the line at infinity, and must therefore intersect in the four poles (real or imaginary) of that line.

This might also be shown as follows: the polar line of any point with respect to the curve, is also its polar line with respect to the polar conic of the point, but if a point be the center of its own polar conic, its polar line with respect to the conic will be the line at infinity, which must therefore be also its polar line with respect to the curve, any such point must consequently be one of the poles of the line at infinity.

If the curve have a double point, since every polar conic passes through it, it is one of these four poles; this might have been anticipated, for the polar conic for such a point is the pair of tangents intersecting (that is, having their center) at the point; if the double point be at infinity, these tangents become a diametral conic, and as every other diametral conic intersects both of them at the double point, it can only meet each of them once again, therefore in this case there can be only *two* other poles; if the curve have a cusp, as all polar conics touch at it, they can only meet each other in two other points; if the cusp be at infinity, its two coincident tangents are a diametral conic, and therefore must contain *all* the poles, but as all the other diametral conics touch this line at the cusp, they can never meet it again; hence in this case the cusp at infinity is the only pole.

Since in general only one point can be found such that its polar conic is an hyperbola having both its asymptotes parallel to fixed directions, it follows that in general only one point can be found the polar conic of which is a circle, since the imaginary asymptotes of every circle pass through fixed imaginary points at infinity. There is but one case that seems to call for any notice, namely, when these circular points at infinity are points upon the curve; from equation (III), which is still true, even though the vectors are now imaginary, we have, supposing ρ_3 the vector to the point at infinity,

$$\frac{1}{\rho_1\rho_2} = 0,$$

which shows that one of the other vectors must be infinite too, that is, that the lines drawn through the pole, parallel to the imaginary asymptotes of the circle, are tangents to the curve at the circular points, and, therefore, that the pole in which they intersect is a double focus.

Hence, *if two of the foci of a curve of the third degree coincide, the polar conic of that point is a circle.*

In considering the subject analytically, I shall use the form of the general equation of the third degree, given by Mr. Salmon,

$$U=0,$$

or $$a_1x^3 + b_2y^3 + c_3z^3 + 6dxyz + 3a_2x^2y + 3a_3zx^2$$
$$+ 3b_3y^2z + 3b_1xy^2 + 3c_1z^2x + 3c_2yz^2 = 0,$$

in which, however, I shall take x and y as the ordinary rectangular coordinates, and z as the line at infinity.

The condition that the polar conic of a point shall be an hyperbola, parabola, or ellipse, is

$$\left(\frac{d^2U}{dxdy}\right)^2 - \frac{d^2U}{dx^2}\frac{d^2U}{dy^2} \gtreqless 0,$$

this can be obtained either directly from the equation of the polar conic, or thus:

The locus of points whose polar conics are hyperbolas, having one asymptote parallel to a fixed direction, has been already proved to be the diameter conjugate to that direction, the equation of which diameter is

$$(\cos\theta)^2\frac{d^2U}{dx^2} + 2\cos\theta\sin\theta\frac{d^2U}{dxdy} + (\sin\theta)^2\frac{d^2U}{dy^2} = 0 \ldots (\text{V}),$$

θ being the angle between the fixed direction and the axis of x.

The form of this equation shows at once that it represents a line which always touches the conic

$$\left(\frac{d^2U}{dxdy}\right)^2 - \frac{d^2U}{dx^2}\frac{d^2U}{dy^2} = 0 \ldots\ldots\ldots\ldots (\text{VI});$$

therefore, since from any point there can be always drawn two tangents (real or imaginary) to any conic, there can be always drawn through any point two diameters to the curve, conjugate to the two values of $(\cos\theta : \sin\theta)$ obtained by solving equation (V).

If ϕ be the angle between the two directions of the line $x \sin\theta - y \cos\theta = 0$ obtained by this solution

$$\tan\phi = \pm \frac{\sqrt{\left\{\left(\frac{d^2U}{dx\,dy}\right)^2 - \frac{d^2U}{dx^2}\frac{d^2U}{dy^2}\right\}}}{\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}} \quad \ldots . \text{(VII)};$$

hence this angle will be real, that is, particular values can then be found for $(\cos\theta : \sin\theta)$ to satisfy (V), or geometrically, the coefficients $(\cos\theta)^2$, $\cos\theta \sin\theta$, $(\sin\theta)^2$ can then be so determined that the diameter may pass through a fixed point, when, for the coordinates of that point,

$$\left(\frac{d^2U}{dx\,dy}\right)^2 - \frac{d^2U}{dx^2}\frac{d^2U}{dy^2} > 0.$$

We thus learn that the conic (VI) divides the plane into two regions, the polar conics of any point in one of which (that from which real tangents to the conic, *i.e.* diameters to the curve, can be drawn) will be an hyperbola, in the other an ellipse, while the polar conic of any point on the conic (VI) itself will be a parabola.

Equation (VII), if ϕ be constant, gives the locus of points whose polar conics will be hyperbolas having a fixed angle between their asymptotes, this locus will always be a conic, except in the one instance when ϕ is a right angle, *i.e.* when the hyperbolas are equilateral, it then becomes the line

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 0.$$

If the coefficients in this equation are each zero, it will be satisfied for any point, in this case every point will have an equilateral hyperbola for its polar conic. Equation (VI) becomes

$$\left(\frac{d^2U}{dx\,dy}\right)^2 + \left(\frac{d^2U}{dx^2}\right)^2 = 0,$$

the equation of two imaginary lines intersecting in the real point

$$\frac{d^2U}{dx\,dy} = 0, \quad \frac{d^2U}{dx^2} = 0;$$

the polar conic of this point should therefore be a parabola, inasmuch as it is a point on (VI), and likewise an equilateral hyperbola, inasmuch as every polar conic with respect to such a curve should be such; furthermore, on actually investigating the conic, it will be found to be neither parabola nor hyperbola, but the line at infinity and another line, its equation being

$$z\left\{2x\left(\frac{d^2U}{dz\,dx}\right)+2y\left(\frac{d^2U}{dy\,dz}\right)+z\left(\frac{d^2U}{dx^2}\right)\right\}=0,$$

x, z being given by the equations

$$(a_1^2+a_2^2)\,x+(a_2a_3+a_1d)\,z=0,$$

$$(a_1^2+a_2^2)\,y+(a_2a_3-a_2d)\,z=0.$$

This apparent contradiction is easily explained; the different species of curves of the second degree are classed *analytically*, not according to their shape or form, but according to the points at infinity through which they pass, each curve is therefore an hyperbola, a parabola, or an ellipse, accordingly as it passes through two real, two coincident, or two imaginary, points at infinity. This analytical classification agrees perfectly with the ordinary ideas of their forms and properties, in all cases but one, that is, when the curve breaks up into the line at infinity and any other line whatsoever, as while it is totally different geometrically from either hyperbola, parabola, or ellipse, it satisfies the analytical definition of them all; since the line of infinity passes through *all* points at infinity, real or imaginary, these two lines are an hyperbola as passing through two real points at infinity, a parabola as touching the line at infinity, an ellipse as passing through two imaginary points at infinity, and a circle as passing through the circular points at infinity.

We can thus see too the reason why, though such a curve should apparently have no double point or point of inflexion whose polar conic is not two lines at right angles to each other, yet that an exception may arise if one of these lines be altogether at infinity.

It cannot however have a real cusp at all, for the only case that this consideration would apparently admit, would be a cusp at infinity, whose tangent was likewise at infinity, but polar conics, with respect to such a curve, we have already shown must be all parabolas.

The conditions among the coefficients, that all polar conics with respect to a curve of the third degree may be equilateral hyperbolas, are

$$a_1+b_1=0,\quad a_2+b_2=0,\quad a_3+b_3=0.$$

And in order that the polar conic of every point may be a parabola, the coefficients in equation (VI) must be each identically zero.

The requisite conditions are

$$a_2^2 - a_1b_1 = 0, \qquad b_1^2 - a_2b_2 = 0,$$
$$d^2 - a_3b_3 = 0, \qquad a_2b_1 - a_1b_2 = 0,$$
$$2b_1d - a_2b_3 - a_3b_2 = 0, \quad 2a_2d - a_1b_3 - a_3b_1 = 0,$$

equivalent to four independent conditions; introducing a quantity k for convenience, these equations give

$$a_2 = ka_1, \quad b_1 = k^2a_1, \quad b_2 = k^3a_1,$$
$$b_3 = k^2a_3, \quad d = ka_3,$$

when these values have been substituted in the general equation it can be easily thrown into the form

$$\{a_1(x+ky) + 3a_3z\}(x+ky)^2 + z^2\{3(c_1x + c_2y) + c_3z\} = 0,$$

which represents a curve having a cusp at infinity to which the line at infinity is tangent, as has been already stated; it is now however shown to be the *only* curve of the third degree possessed of this property.

The point whose polar conic is a circle, is the intersection of the two lines

$$\frac{d^2U}{dxdy} = 0, \quad \frac{d^2U}{dx^2} - \frac{d^2U}{dy^2} = 0 \ldots\ldots\ldots (\text{VIII}),$$

these equations may be got, either directly as the conditions that the equation of the conic may represent a circle, or as the real intersection of the imaginary diameters, conjugate to the directions of the circular points at infinity, the equations of which are

$$\frac{d^2U}{dx^2} - \frac{d^2U}{dy^2} \pm 2\sqrt{(-1)}\,\frac{d^2U}{dxdy} = 0.$$

Should either of the two equations (VIII) be satisfied identically, the other then becomes a locus the polar conic of any point on which will be a circle. In this case the conic (VI) breaks up into two right lines, through the intersection of which the present locus passes, the polar conic of this point of intersection presents the same apparent anomaly, to be explained in the same way, as that noticed above.

Any line taken at random will be a polar line having four, or in the general case of a curve of the n^{th} degree,

$(n-1)^2$ fixed poles, but every conic will not be a polar conic; as in the case of polar conics two points fix the pole, but through these two points there can be made to pass an infinite number of other conics, none of which will be polar conics. As a simple illustration take the case of a curve of the third degree having a double point, every polar conic must pass through it, therefore no conic not passing through it can be a polar conic at all.

As five conditions are required to determine a conic, that a conic passing through two fixed points may be a polar conic, three other conditions must be fulfilled equivalent to demanding, in the case of a curve of the third degree, that the polar lines of three other points on the conic may pass through the intersection of the polar lines of the two fixed points.

Several of the foregoing theorems can be easily extended to the general case of a curve of the n^{th} degree, but as (VI) would then be a curve of the $2(n-2)^{\text{th}}$ degree, the consideration would, except in particular cases, not be so interesting.

Crossmaylen,
9th November, 1857.

ON THE EQUATION OF THE SURFACE OF CENTRES OF AN ELLIPSOID.

By the Rev. George Salmon.

Having lately worked out the equation of this surface, I propose to print it here, together with a short sketch of the method by which it was obtained.

Let the axes of the given ellipsoid be a, b, c, and let the major axes of the two confocal surfaces, which can be drawn through any point on the surface, be a', a'', then I start with the principle (which can easily be proved, see *Cambridge and Dublin Math. Jour.*, Vol. v, p. 177) that the centres of the two circles of greatest and least curvature corresponding to that point, are the poles of the tangent plane to the given surface with respect to the two confocal surfaces. In other words, the coordinates of one of these centres are

$$x=\frac{a'^2x'}{a^2},\quad y=\frac{b'^2y'}{b^2},\quad z=\frac{c'^2z'}{c^2}\,\ldots\ldots\ldots\ldots(A).$$

From these equations we can at once find the locus of the centres corresponding to a line of curvature on the given surface. For, a' being constant along a line of curvature, and $x'y'z'$ satisfying the relations

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1,\quad \frac{x'^2}{a^2a'^2}+\frac{y'^2}{b^2b'^2}+\frac{z'^2}{c^2c'^2}=0,$$

we have by substitution from the former equations (and at the same time writing $a'^2=a^2-h^2$, $b'^2=b^2-h^2$, $c'^2=c^2-h^2$)

$$\frac{a^2x^2}{(a^2-h^2)^2}+\frac{b^2y^2}{(b^2-h^2)^2}+\frac{c^2z^2}{(c^2-h^2)^2}=1,$$

$$\frac{a^2x^2}{(a^2-h^2)^3}+\frac{b^2y^2}{(b^2-h^2)^3}+\frac{c^2z^2}{(c^2-h^2)^3}=0.$$

Equations which represent a curve of the fourth degree. And if between those equations we eliminate h, we shall have the equation of the surface of centres. This elimination however being between two equations, each of the sixth degree in h^2, is so laborious as to be scarcely practicable. It can easily be imagined, however, that the conception of the surface of centres, as the aggregate of the curves corresponding to lines of curvature on the original surface, may not give rise to the simplest mode of generating that surface. And accordingly the method that I proceed now to explain, instead of leading to elimination between two equations of the sixth degree, only leads to elimination between an equation of the second and one of the third degree. Substitute in the equation (A) the expressions for x', y', z' in terms of the axes of the confocal surfaces, viz.

$$x'^2=\frac{a^2a'^2a''^2}{(a^2-b^2)(a^2-c^2)},\quad y'^2=\frac{b^2b'^2b''^2}{(b^2-c^2)(b^2-a^2)},\quad z'^2=\frac{c^2c'^2c''^2}{(c^2-a^2)(c^2-b^2)},$$

and we get

$$a^2x^2=\frac{a'^6a''^2}{(a^2-b^2)(a^2-c^2)},\quad b^2y^2=\frac{b'^6b''^2}{(b^2-c^2)(b^2-a^2)},\quad c^2z^2=\frac{c'^6c''^2}{(c^2-a^2)(c^2-b^2)},$$

or if $\qquad h^2=a^2-a'^2,\quad k^2=a^2-a''^2,$

$$a^2x^2=\frac{(a^2-h^2)^3(a^2-k^2)}{(a^2-b^2)(a^2-c^2)},\quad b^2y^2=\frac{(b^2-h^2)^3(b^2-k^2)}{(b^2-c^2)(b^2-a^2)},\quad c^2z^2=\frac{(c^2-h^2)^3(c^2-k^2)}{(c^2-a^2)(c^2-b^2)}$$

...............(B).

Now if we form the biquadratic equation, three of those roots are equal to h^2 and one equal to k^2,

$$\theta^4 - P\theta^3 + Q\theta^2 - R\theta + S = 0,$$

it is manifest that the three equations (B) (which give the results of substituting a, b, c for θ) give the linear relations between P, Q, R, S, enabling us to express any three of those quantities in terms of the remaining one. But since these quantities are coefficients of a biquadratic equation having three roots equal, P, Q, R, S are connected by the known relations obtained by equating to zero the two invariants of the equation, or we have

$$\left.\begin{aligned} 12S - 3PR + Q^2 = 0 \\ 32QS + PQR - 9R^2 - 9P^2S = 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (C),$$

one of which gives rise to an equation of the second, the other to one of the third degree.

Let $a^2 + b^2 + c^2 = p$, $a^2b^2 + b^2c^2 + c^2a^2 = q$, $a^2b^2c^2 = r$,

then I have found it convenient to express Q, R, S in terms of $P - p$ which I shall call λ, and the equations (B) will be found to give rise to the relations

$$\begin{aligned} P &= \lambda + p, \\ Q &= p\lambda + q + \phi, \\ R &= q\lambda + r + \psi, \\ S &= r\lambda + \rho, \end{aligned}$$

where $\phi = a^2x^2 + b^2y^2 + c^2z^2$,

$$\psi = a^2(b^2 + c^2)x^2 + b^2(c^2 + a^2)y^2 + c^2(a^2 + b^2)z^2,$$

$$\rho = r(x^2 + y^2 + z^2).$$

Substituting these values in the equations (C), the question is reduced to the elimination of λ between

$$A\lambda^2 + (2p\phi - 3\psi - B)\lambda + \phi^2 + 2q\phi - 3p\psi + 12\rho + C = 0,$$

$$\begin{aligned} &B\lambda^3 + (p\psi + q\phi - 9\rho + pB - 8C)\lambda^2 \\ &+ \{\phi\psi + (pq + 33r)\phi + (p^2 - 17q)\psi + 14p\rho + qB - 8rA\}\lambda \\ &+ p\phi\psi + 32\phi\rho - 9\psi^2 + pr\phi + (pq - 18r)\psi + (32q - 9p^2)\rho + rB = 0, \end{aligned}$$

where $A = p^2 - 3q$, $B = pq - 9r$, $C = q^2 - 3pr$.

We might, if we chose, by subtracting the first equation multiplied by $B\lambda$ from the second multiplied by A, reduce the problem to elimination between two quadratics, and consequently obtain the result in the form $U^2 = VW$, where U is of the sixth, V of the fourth, and U of the eighth degree in x, y, z. As this method, however, introduces a new constant factor, I have preferred, instead of calculating the equation in this form, to obtain it in the expanded form. I have used the following abbreviation:

$$ax = \xi,\quad by = \eta,\quad cz = \zeta,\quad b^2 - c^2 = \alpha,\quad c^2 - a^2 = \beta,\quad a^2 - b^2 = \gamma,$$

and in order to make apparent a certain symmetry between the plane at infinity and the three principal planes, I write $\omega^2 = -1$. The result then is as follows:

$$\alpha^6\xi^{12} + \beta^6\eta^{12} + \gamma^6\zeta^{12} + \alpha^6\beta^6\gamma^6\omega^{12}$$

$$+ 3(\beta^2 + \gamma^2)\{\beta^4\eta^{10}\zeta^2 + \gamma^4\zeta^{10}\eta^2 + \alpha^6\xi^{10}\omega^2 + \alpha^6\beta^4\gamma^4\xi^2\omega^{10}\}$$

$$+ 3(\gamma^2 + \alpha^2)\{\gamma^4\zeta^{10}\xi^2 + \alpha^4\xi^{10}\zeta^2 + \beta^6\eta^{10}\omega^2 + \alpha^4\beta^6\gamma^4\eta^2\omega^{10}\}$$

$$+ 3(\alpha^2 + \beta^2)\{\alpha^4\xi^{10}\eta^2 + \beta^4\eta^{10}\xi^2 + \gamma^6\zeta^{10}\omega^2 + \alpha^4\beta^4\gamma^6\zeta^2\omega^{10}\}$$

$$+ 3(\beta^4 + 3\beta^2\gamma^2 + \gamma^4)\{\beta^2\zeta^4\eta^8 + \gamma^2\eta^4\zeta^8 + \alpha^6\xi^8\omega^4 + \beta^2\gamma^2\omega^8\xi^4\}$$

$$+ 3(\gamma^4 + 3\gamma^2\alpha^2 + \alpha^4)\{\gamma^2\xi^4\zeta^8 + \alpha^2\zeta^4\xi^8 + \beta^6\eta^8\omega^4 + \gamma^2\alpha^2\omega^8\eta^4\}$$

$$+ 3(\alpha^4 + 3\alpha^2\beta^2 + \beta^4)\{\alpha^2\eta^4\xi^8 + \beta^2\xi^4\eta^8 + \gamma^6\zeta^8\omega^4 + \alpha^2\beta^2\omega^8\zeta^4\}$$

$$+ 3(2\alpha^4 + 3\alpha^2\beta^2 + 3\alpha^2\gamma^2 - 7\beta^2\gamma^2)$$
$$\times\{\alpha^2\xi^8\eta^2\zeta^2 + \beta^4\eta^8\xi^2\omega^2 + \gamma^4\zeta^8\xi^2\omega^2 + \beta^4\gamma^4\alpha^2\omega^8\eta^2\zeta^2\}$$

$$+ 3(2\beta^4 + 3\beta^2\gamma^2 + 3\beta^2\alpha^2 - 7\gamma^2\alpha^2)$$
$$\times\{\beta^2\eta^8\zeta^2\xi^2 + \gamma^4\zeta^8\eta^2\omega^2 + \alpha^4\xi^8\eta^2\omega^2 + \gamma^4\alpha^4\beta^2\omega^8\zeta^2\xi^2\}$$

$$+ 3(2\gamma^4 + 3\gamma^2\alpha^2 + 3\gamma^2\beta^2 - 7\alpha^2\beta^2)$$
$$\times\{\gamma^2\zeta^8\xi^2\eta^2 + \alpha^4\xi^8\zeta^2\omega^2 + \beta^4\eta^8\zeta^2\omega^2 + \alpha^4\beta^4\gamma^2\omega^8\xi^2\eta^2\}$$

$$+ (\beta^6 + \gamma^6 + 9\beta^4\gamma^2 + 9\beta^2\gamma^4)(\eta^6\zeta^6 + \alpha^6\xi^6\omega^6)$$

$$+ (\gamma^6 + \alpha^6 + 9\gamma^4\alpha^2 + 9\gamma^2\alpha^4)(\zeta^6\xi^6 + \beta^6\eta^6\omega^6)$$

$$+ (\alpha^6 + \beta^6 + 9\alpha^4\beta^2 + 9\alpha^2\beta^4)(\xi^6\eta^6 + \gamma^6\zeta^6\omega^6)$$

$$+ 3\{\alpha^6 + 6\alpha^4\beta^2 + 3\alpha^4\gamma^2 + 3\alpha^2\beta^4 + \beta^4\gamma^2 - 21\alpha^2\beta^2\gamma^2\}$$
$$\times\{\xi^6\eta^4\zeta^2 + \beta^2\eta^6\xi^4\omega^2 + \gamma^4\zeta^6\xi^2\omega^4 + \beta^2\gamma^4\eta^2\zeta^4\omega^6\}$$

$$+ 3\{\alpha^6 + 6\alpha^4\gamma^2 + 3\alpha^4\beta^2 + 3\alpha^2\gamma^4 + \beta^2\gamma^4 - 21\alpha^2\beta^2\gamma^2\}$$
$$\times\{\xi^6\eta^2\zeta^4 + \gamma^2\zeta^6\xi^4\omega^2 + \beta^4\eta^6\xi^2\omega^4 + \beta^4\gamma^2\eta^4\zeta^2\omega^6\}$$

$$+3\{\beta^6+6\beta^4\alpha^2+3\beta^4\gamma^2+3\beta^2\alpha^4+\alpha^4\gamma^2-21\alpha^2\beta^2\gamma^2\}$$
$$\times\{\eta^6\xi^4\zeta^2+\alpha^2\xi^6\eta^4\omega^2+\gamma^4\zeta^6\eta^2\omega^4+\alpha^2\gamma^4\xi^2\zeta^4\omega^6\}$$
$$+3\{\beta^6+6\beta^4\gamma^2+3\beta^4\alpha^2+3\beta^2\gamma^4+\alpha^2\gamma^4-21\alpha^2\beta^2\gamma^2\}$$
$$\times\{\eta^6\xi^2\zeta^4+\gamma^2\zeta^6\eta^4\omega^2+\alpha^4\xi^6\eta^2\omega^4+\alpha^4\gamma^2\xi^4\zeta^2\omega^6\}$$
$$+3\{\gamma^6+6\gamma^4\alpha^2+3\gamma^4\beta^2+3\gamma^2\alpha^4+\beta^2\alpha^4-21\alpha^2\beta^2\gamma^2\}$$
$$\times\{\zeta^6\xi^4\eta^2+\alpha^2\xi^6\zeta^4\omega^2+\beta^4\eta^6\zeta^2\omega^4+\alpha^2\beta^4\xi^2\eta^4\omega^6\}$$
$$+3\{\gamma^6+6\gamma^4\beta^2+3\gamma^4\alpha^2+3\gamma^2\beta^4+\alpha^2\beta^4-21\alpha^2\beta^2\gamma^2\}$$
$$\times\{\zeta^6\xi^2\eta^4+\beta^2\eta^6\zeta^4\omega^2+\alpha^4\xi^6\zeta^2\omega^4+\alpha^4\beta^2\xi^4\eta^2\omega^6\}$$
$$-3\{14(\alpha^4\beta^2+\alpha^2\beta^4+\beta^4\gamma^2+\beta^2\gamma^4+\gamma^4\alpha^2+\gamma^2\alpha^4)+20\alpha^2\beta^2\gamma^2\}$$
$$\times\{\alpha^2\xi^4+\beta^2\eta^4+\gamma^2\zeta^4+\alpha^2\beta^2\gamma^2\omega^4\}\,\xi^2\eta^2\zeta^2\omega^2$$
$$+9\{\alpha^4\beta^2+\alpha^2\beta^4+\beta^4\gamma^2+\beta^2\gamma^4+\gamma^4\alpha^2+\gamma^2\alpha^4-14\alpha^2\beta^2\gamma^2\}$$
$$\times\{\xi^4\eta^4\zeta^4+\alpha^2\beta^2\xi^4\eta^4\omega^4+\beta^2\gamma^2\eta^4\zeta^4\omega^4+\gamma^2\alpha^2\zeta^4\xi^4\omega^4\}$$
$$-3\{4\alpha^8-7\alpha^6(\beta^2+\gamma^2)-198\alpha^4\beta^2\gamma^2+68\alpha^2\beta^2\gamma^2(\beta^2+\gamma^2)+42\beta^4\gamma^4\}$$
$$\times\{\eta^2\zeta^2+\alpha^2\xi^2\omega^2\}\,\xi^2\eta^2\zeta^2\omega^2$$
$$-3\{4\beta^8-7\beta^6(\gamma^2+\alpha^2)-198\beta^4\gamma^2\alpha^2+68\alpha^2\beta^2\gamma^2(\gamma^2+\alpha^2)+42\gamma^4\alpha^4\}$$
$$\times\{\zeta^2\xi^2+\beta^2\eta^2\omega^2\}\,\xi^2\eta^2\zeta^2\omega^2$$
$$-3\{4\gamma^8-7\gamma^6(\alpha^2+\beta^2)-198\gamma^4\alpha^2\beta^2+68\alpha^2\beta^2\gamma^2(\alpha^2+\beta^2)+42\alpha^4\beta^4\}$$
$$\times\{\xi^2\eta^2+\gamma^2\zeta^2\omega^2\}\,\xi^2\eta^2\zeta^2\omega^2.$$

The sections of the surface by the three principal planes, are an ellipse three times, and the evolute of an ellipse. Thus, if we make $z=0$ in the equation, we get

$$\{\alpha^2a^2x^2+\beta^2b^2y^2-\alpha^2\beta^2\}^3\{(a^2x^2+b^2y^2-\gamma^2)^3+27a^2b^2\gamma^2x^2y^2\}.$$

But it appears also that the section by the plane at infinity is also of the same nature, since the highest terms of the equation are

$$(a^2x^2+b^2y^2+c^2z^2)^3\{(a^2\alpha^2x^2+b^2\beta^2y^2+c^2\gamma^2z^2)^3-27a^2b^2c^2\alpha^2\beta^2\gamma^2x^2y^2z^2\}.$$

In conclusion, I may mention that the reciprocal of this surface is one of the fourth degree. The equation of the reciprocal was, I believe, first given by Dr. Booth in his tract on *Tangential Coordinates*. The tangential coordinates employed by him are the reciprocals of the intercepts made by the tangent plane on the axes, and these are evidently proportional to the ordinary coordinates of the reciprocal surface. It is easy to see that the tangent plane to one of confocal surfaces through the point $x'y'z'$ is also a tangent plane to the surface of centres. The reciprocals of the intercepts which this plane makes on the axes are therefore

$$\xi = \frac{x'}{a'^2}, \quad \eta = \frac{y'}{b'^2}, \quad \zeta = \frac{z'}{c'^2}.$$

The relation

$$\frac{x'^2}{a^2a'^2} + \frac{y'^2}{b^2b'^2} + \frac{z'^2}{c^2c'^2} = 0$$

gives

$$(\xi^2 + \eta^2 + \zeta^2) = (a^2 - a'^2)\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2}\right),$$

and the relation

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} = 1$$

gives

$$(a^2\xi^2 + b^2y^2 + c^2\zeta^2 - 1) = (a^2 - a'^2)(\xi^2 + \eta^2 + \zeta^2).$$

The equation of the reciprocal is therefore

$$(\xi^2 + \eta^2 + \zeta^2)^2 = (a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 - 1)\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2}\right).$$

Nov. 20, 1857.

ON THE FORMATION OF TABLES OF LOGARITHMS OF THE TRIGONOMETRICAL RATIOS.

By H. W. ELPHINSTONE, M.A., Trinity College, Cambridge.

THE following account of the method of forming tables of the logarithms of the trigonometrical functions is taken from Francœur's *Cours Complet de Mathematiques Pures*, Art. 588. This method is not given in any of the books in common use at this University, and I have therefore been led to insert it in the *Journal*, in the belief that it might be of use to students.

Suppose that angles in the tables have a constant difference of 10″, let k = circular measure of 10″, and if θ be any angle given in the tables let $\theta = nk$. We have, by Euler's formula,

$$\begin{aligned}
\sin\theta &= \sin nk \\
&= nk\left\{1 - \frac{n^2k^2}{1^{3|1}} + \frac{n^4k^4}{1^{5|1}} - \&c.\right\}, \\
\cos\theta &= \quad 1 - \frac{n^2k^2}{1^{2|1}} + \frac{n^4k^4}{1^{4|1}} - \&c.
\end{aligned}$$

Let

$$\begin{aligned}
y &= \frac{n^2k^2}{1^{3|1}} - \frac{n^4k^4}{1^{5|1}} + \&c., \\
z &= \frac{n^2k^2}{1^{2|1}} - \frac{n^4k^4}{1^{4|1}} + \&c.
\end{aligned}$$

Then we shall have

$$\begin{aligned}
\sin\theta &= nk(1-y), \\
\cos\theta &= 1 - z,
\end{aligned}$$

and taking logarithms to base 10,

$$\log\sin\theta = \log n + \log k - M\left\{y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\right\},$$

$$\log\cos\theta = -M\left\{z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots\right\},$$

or substituting for y and z their values

$$\log\sin\theta = \log n + \log k - \frac{Mk^2}{2.3}n^2 - \frac{Mk^4}{4.5.9}n^4 - \&c.,$$

$$\log\cos\theta = -\frac{Mk^2}{2}n^2 - \frac{Mk^4}{3.4}n^4 - \&c.,$$

writing these under the form

$$\log\sin\theta = \log n + \log k - An^2 - Bn^4 - \&c.,$$

$$\log\cos\theta = -A_1n^2 - B_1n^4 - \&c.,$$

the values of A, B, A_1, B_1 admit of easy calculation by means of a table of logarithms, they are found to be as follows:

$$\begin{aligned}
\log A &= \overline{10}.2307828 \\
\log B &= \overline{20}.1248113 \\
\log A_1 &= \overline{10}.7079041 \\
\log B_1 &= \overline{19}.3009025
\end{aligned}$$

If we do not apply these series to the calculation of log sines and log cosines of angles greater than 12° and require 9 decimal places only, we may stop at the terms involving n^4. The logarithms of tangents, cotangents, secants, and cosecants are obtained from those of sines and cosines by addition only.

Although we *may* apply this method so long as the angle is not greater than 12°, it is more convenient to apply the following method when the angle is greater than 5°:

$$\frac{\sin(\theta+k)}{\sin\theta} = \frac{\sin\theta\cos k + \cos\theta\sin k}{\sin\theta}$$

$$= \cos k\,(1+\cot\theta\tan k)$$

$$= 1 + k\cot\theta \text{ (true to 9 decimal places)},$$

$$\log\sin(\theta+k) - \log\sin\theta = \log(1+k\cot\theta)$$

$$= \quad Mk\cot\theta = \Delta \text{ suppose.}$$

Similarly

$$\log\cos(\theta+k) - \log\cos\theta = -Mk\tan\theta = \Delta' \text{ suppose,}$$

where Δ, Δ' are the successive tabular differences. The values of Δ, Δ' may easily be calculated by means of a table of common logarithms; as they change but slowly, the same values may be used for the computation of several successive sines and cosines.

By means of these formulas we can calculate sines and cosines and therefore all the other trigonometrical functions of angles from 0° up to 45°, and there is no need to proceed any farther.

The following formula of verification is of use to enable us to see whether we have allowed any errors to accumulate:

$$\log 2 + \log\sin\theta + \log\cos\theta = \log\sin 2\theta.$$

If we add 10 to the characteristic of the logarithms thus found we shall have the "tabular" logarithms.

For example let us calculate sin(4°.30′).

Here $n = 1620$,

$$\log A = \overline{10}.2307828 \qquad \log B = \overline{20}.1248113$$

$$\log n^2 = 6.4190300 \qquad \log n^4 = 12.8380600$$

$$\overline{4}.6498128 \qquad \overline{8}.9628713$$

The corresponding numbers are 0.00044649

and 0.00000009

hence we have

$$\log k = \bar{5}.68557487$$
$$\log n = 3.20951501$$
$$-.00044649$$
$$-.00000009$$
$$\bar{2}.89464330 = \log \sin 4^\circ.30' \text{ to base } 10$$
$$\text{and } 8.89464330 = \text{tabular } \log \sin 4^\circ.30'.$$

Again, let us obtain the tabular difference between $\log \sin 10^\circ.10'.30''$ and $\log \sin 10^\circ.10'.40''$.

$$\log \cot \theta = 0.7459888$$
$$\log (Mh) = \bar{5}.3233592$$
$$\log \Delta = \bar{4}.0693480$$
$$\Delta = 0.00011731.$$

AN EXAMPLE OF THE INSTINCT OF CONSTRUCTIVE GEOMETRY.

By Cecil James Monro, Trinity College, Cambridge.

There is a class of optical illusions, as they may be called, arising from a readiness in the eye to detect any simple geometrical relations which may exist among an assemblage of points presented to it. Curves of perhaps considerable complexity, and apparently almost of arbitrary form, are often as it were forced upon it; but of course always reducible to exact geometrical laws.

Suppose two systems of lines (I will suppose them, for the sake of distinctness rather of language than conception, to lie on the same surface, and that plane), varying respectively by finitely differing values of a single parameter, to intersect; as many systems of polygons may be drawn through the intersections of corresponding pairs, as ingenuity may conceive relations between the parameters in order to determine correspondence. But, in the case of lines actually drawn, the eye is irresistibly led to select two of these relations, and to form the appropriate polygon, or rather the simplest curve that can be drawn through its determining points. The principle upon which we instinctively proceed is to step from one point to the nearest which would not require a violent deviation from a direction previously chosen: thus, if M, N, M', N', M'', N'' be consecutive pairs of lines, we shall proceed from (mn) to $(m'n')$, and thence to $(m''n'')$ (to designate the points of intersection by an obvious notation), or from (mn'') to $(m'n')$, and thence to $(m''n)$, and so on. The relations between the parameters which will prescribe these two courses are found by equating their sum and difference respectively to a constant quantity, which is itself the parameter of the new system. From either equation and those of the given systems the first parameters may be eliminated, and an equation results satisfied by the coordinates of the new series of points, which is in fact the equation of the simplest line that can be drawn through them. Perhaps there are short-sighted persons (for they seem to be quickest at detecting these relations) who can see the curves formed by proceeding by two steps of the one system to one of the other, or generally any of the curves depending on the relation (m and n being the parameters)

$$Am + Bn = C;$$

but I can only catch those given by $A \pm B = 0$, and therefore

$$m \pm n = C.$$

I will now indicate a few examples which will be easily verified, as they may be observed every day.

1. If each system consist of radii passing through a fixed point, of which each makes the same angle with that which follows it, then, in the case in which this angle is the same for both systems, the resulting systems are circles and hyperbolæ through the fixed points. The first is evident from Euclid's third book: for the second, take the line joining the points for the axis of x, and the line perpendicular to this half way between the points for that of y; let the distance of the two points be $2a$, and the interval between consecutive radii α. Then

$$\frac{y}{x-a} = \tan m\alpha,$$

$$\frac{y}{x+a} = \tan n\alpha;$$

and, if m and n be eliminated by means of the equation

$$m + n = \frac{\omega}{\alpha},$$

ω being an arbitrary constant,

$$x^2 - y^2 - 2\cot\omega . xy - a^2 = 0;$$

giving a hyperbola passing through the points, and having for its asymptotes the lines through the symmetrical point which make with the line joining the points angles whose tangents are equal to

$$\tfrac{1}{2}(-\cot\omega \pm \operatorname{cosec}\omega).$$

An example may be observed in the figure illustrating the reflexion of a conical pencil of rays at a plane surface in Goodwin's *Course*.

2. Taking a pair of series of concentric circles whose radii are in some regular progression, we have ovals, varying of course in form according to the nature of the progression. If the progressions are arithmetical the ovals are those of Cassini; and when the successive differences are the same in both, they become confocal ellipses and hyperbolæ.

If the progressions are geometrical, so that

$$\rho = ak^m,$$
$$\rho' = bk^n,$$
$$\rho\rho'^{\pm 1} = ab^{\pm 1}k^C = c^2 \text{ or } \gamma,$$

the two systems are, first, the curves of the fourth degree,

$$(r^2 + a^2)^2 - 4a^2r^2\cos^2\theta = c^4,$$

and, secondly, the circles,

$$r^2 + 2a\frac{1+\gamma}{1-\gamma}r\cos\theta + a^2 = 0.$$

If the radii vary as some power (the k^{th}) of the number representing the order of the circle, we get

$$\left(\frac{\rho}{a}\right)^{\frac{1}{k}} \pm \left(\frac{\rho'}{b}\right)^{\frac{1}{k}} = C,$$

and find new forms of ovals, including the simpler case of curves of the second order and even straight lines when $k = \frac{1}{2}$.

To this general class belong the two systems observed to result from the intersections of the waves produced by throwing two stones into water. Until the waves have spread considerably the resulting curves are scarcely distinguishable from the ellipses and hyperbolæ given by the case of arithmetical progression.

3. If the given lines are not in the same plane or in planes at all, the result is of course the same, and it is simplest in practice to consider their projections. Thus the apparent intersections of two rows of railings, each row in a vertical plane, and each rail parallel to those in its own row, lie in curves of the second order, when projected on a plane: (a case mentioned in a paper referred to in the note).*

And generally, in the case of straight lines, if one system be defined by two equations containing m in the p^{th} and p'^{th} degrees respectively, and the other by two containing n in the q^{th} and q'^{th} degrees; the visual cone directed by the locus of apparent intersections is of a degree not higher than the $(p+p'+1)(q+q'+1)^{\text{th}}$. For if $u, v, w, \xi, \eta, \zeta, x, y, z$ be the coordinates of the eye, a point in a line of one system, and a point of the cone required; ξ, η, ζ satisfy the first pair of equations and their coefficients contain m: at the same time they are functions of the first degree of the quantity expressed by the equal ratios

$$\frac{\xi-u}{x-u}, \quad \frac{\eta-v}{y-v}, \quad \frac{\zeta-w}{z-w};$$

they may therefore be eliminated by substitution in the two equations, which will thus be of the first degree in terms of this quantity, and of the p^{th} and p'^{th} in terms of m. When this quantity is eliminated m enters into the resulting equation in the $(p+p')^{\text{th}}$ degree. In the same way the corresponding equation for the other system contains n in the $(q+q')^{\text{th}}$ degree. But they each contain x, y, z in the first degree and are actually therefore of the degrees $p+p'+1$ and $q+q'+1$ in terms of their variables. Further the equation, $m \pm n =$ a constant, is of the first degree. The result therefore of elimination is of a degree not higher than the $(p+p'+1)(q+q'+1)^{\text{th}}$ in terms of x, y, z.

4. Take the example of parallel circular equidistant rings lying in a right circular cylinder. Let the axis of the cylinder be the axis of y, and the eye be in the axis of z at a distance c; the radius being a. The nearer sides of the rings will apparently intersect the further; let us find the result projected on the plane xy. If $a\cos\theta$, η, and $a\sin\theta$ be the coordinates of a point in a ring,

$$a\cos\theta = \frac{\eta x}{y}, \quad a\sin\theta = \frac{c(y-\eta)}{y},$$

* Gergonne, t. XIX. pp. 371-374. Note sur la théorie Anal. du Moiré.

whence $$(x^2+c^2)\eta^2-2c^2y\eta+(c^2-a^2)y^2=0;$$

now any pair of simultaneous values of x and y gives two values of η, that is to say, determines the apparent intersection of two rings; and a value of η being the parameter for each ring, we have to form the equations

$$\eta_1\pm\eta_2=\text{a constant};$$

that is, h and k being arbitrary parameters,

$$hx^2-2c^2y+hc^2=0,$$

and $$(c^2-a^2)x^2y^2+k^4x^2-c^2a^2y^2+k^4c^2=0.$$

An example is furnished by a cylinder of wire gauze.

Some apparent examples owe their vividness to another cause. The circles and hyperbolæ of the first case will be produced by two radii revolving, each about one point, at the same rate, the relation between the parameters being now not selected by the eye but prescribed by the circumstances of the motion: and here we have the further advantage of perfect continuity. The effect is heightened by increasing the number of revolving lines, as the impression on the retina may be thus renewed before it has time to fade. Hence the circles are seen in great perfection on looking through one wheel of a carriage in motion at the opposite one. If the wheels turned round in opposite ways we should see the hyperbolæ.

NOTE ON THE EXPANSION OF THE TRUE ANOMALY.

By A. Cayley.

If the true anomaly and the mean anomaly are respectively denoted by u, m, and if e be the excentricity, then as usual $u-e\sin u=m$; and if we write $\lambda=\dfrac{1-\sqrt{(1-e^2)}}{e}$ and take c to denote the base of the hyperbolic system of logarithms, we have

$$u=m+2\Sigma_1^\infty A_r\frac{\sin rm}{r},$$

and $$A_r=\lambda^r c^{-\frac{1}{2}re(\lambda-\lambda^{-1})}+\lambda^{-r}c^{\frac{1}{2}re(\lambda-\lambda^{-1})},$$

where, after expanding the exponentials, the negative powers of λ are to be rejected and the term independent of λ is to be multiplied by $\frac{1}{2}$ (see *Camb. Math. Journal*, t. I. p. 228 and t. III. p. 165).

It is easily seen that e^r is the lowest power of e which enters into the value of A_r and the question arises to find the numerical coefficient of the term in question; this is readily obtained from the formula; in fact considering first a term of the form

$$\lambda^{-r} e^s (\lambda - \lambda^{-1})^s,$$

since λ is itself of the order e, when the negative powers of λ are rejected this is at least of the order e^s and it is consequently to be neglected if $s > r$. But if $s < r$ all the powers of λ are negative and the term is to be rejected. The only case to be considered is therefore that of $s = r$, in which case there is a term containing e^r. We thus obtain from $\lambda^{-r} c^{\frac{1}{2}re(\lambda-\lambda^{-1})}$ the term

$$\frac{1}{2}\, \frac{r^r e^r}{2^r\, 1.2.3\ldots r}.$$

In the next place a term of the form $\lambda^r e^s (\lambda - \lambda^{-1})^s$ is at least of the order e^s if $s > r$, or the terms to be considered are those for which $s =$ or $< r$. But in such term the only part of the order e^r is

$$(-)^s \lambda^{r-s} e^s,$$

or since neglecting higher powers of e we have $\lambda = \frac{1}{2}e$, this is

$$(-)^s\, 2^{-r+s}\, e^r,$$

and the set of terms arising from

$$\lambda^r\, c^{-\frac{1}{2}re(\lambda-\lambda^{-1})},$$

is
$$\frac{e^r}{2^r}\left\{1 + \frac{r}{1} + \frac{r^2}{1.2} \ldots + \frac{r^{r-1}}{1.2\ldots(r-1)} + \frac{1}{2}\,\frac{r^r}{1.2\ldots r}\right\},$$

the last term being divided by 2 because arising from a term independent of λ. Hence the first term of A_r is

$$\frac{e^r}{2^r}\left\{1 + \frac{r}{1} + \frac{r^2}{1.2} \ldots + \frac{r^r}{1.2\ldots r}\right\},$$

a result which it may be remarked is contained in the general formula given in Hansen's Memoir "*Entwickelung des Products &c.*," Leipzig Trans., t. II. p. 277 (1853).

The preceding expression is

$$= \frac{e^r c^r}{2^r}\, \frac{1}{\Gamma(r+1)} \int_r^\infty x^r c^{-x} dx,$$

and to find its value when r is large, we have

$$\int_r^\infty x^r c^{-x} dx = \int_0^\infty (y+r)^r e^{-y-r} dy = r^r c^{-r} \int_0^\infty \left(1+\frac{y}{r}\right)^r e^{-y} dy$$

$$= r^r c^{-r} \int_0^\infty c^{-y+r\log\left(1+\frac{y}{r}\right)} dy$$

$$= r^r c^{-r} \int_0^\infty c^{-\frac{y^2}{2r}+\frac{y^3}{3r^2}-\&c.} dy$$

$$= r^r c^{-r} \int_0^\infty \left(1+\frac{y^3}{3r^2}+\ldots\right) e^{-\frac{y^2}{2r}} dy$$

$$= r^r c^{-r} \sqrt{2r} \int_0^\infty \left\{1+\frac{2\sqrt{2}}{3\sqrt{r}} z^3+\ldots\right) e^{-z^2} dz,$$

or neglecting all the terms except the first, this is

$$= r^r c^{-r} \sqrt{2r} \int_0^\infty e^{-z^2} dz$$

$$= \sqrt{2\pi r}\, r^r c^{-r}.$$

Hence multiplying by $\frac{1}{2^r} e^r c^r \frac{1}{\Gamma(r+1)}$ and observing that when r is large, we have, by a well known formula,

$$\Gamma(r+1) = \sqrt{2\pi r}\, r^r c^{-r},$$

we obtain finally the result that when r is large the first term of A_r is approximately

$$= \left(\frac{ec}{2}\right)^r.$$

I take the opportunity of mentioning the following somewhat singular theorem, which seems to belong to a more general theory: viz. if $u - e\sin u = m$, then we have

$$\log(1-e\cos u) = \frac{1}{\alpha}\log(1-\alpha e\cos\phi),$$

where $$\phi - \frac{1}{\alpha}\tan\phi = m,$$

provided that the negative powers of α are rejected and α is then put equal to unity.

To shew this, we have by Lagrange's theorem, observing that

$$\frac{d}{dm}F(1-e\cos m)=e\sin m F'(1-e\cos m),$$

$$F(1-e\cos u)=F(1-e\cos m)+\frac{e^2}{1}\sin^2 m F'(1-e\cos m)$$
$$+\frac{e^3}{1.2}\frac{d}{dm}\sin^3 m F'(1-e\cos m)+\&c.,$$

and the coefficient of e^r in $F(1-e\cos u)$ is

$$\frac{(-)^r}{1.2\ldots(r-1)}\left\{\frac{1}{r}F_r\cos^r m+\frac{r-1}{1}F_{r-1}\cos^{r-2}m\sin^2 m\right.$$
$$\left.-\frac{(r-1)(r-2)}{1.2}F_{r-2}\frac{d}{dm}(\cos^{r-3}m\sin^3 m)+\&c.\right\},$$

where $F_r=F^r(1)$.

Hence in particular when $Fx=\log x$, $F_r=(-)^{r-1}1.2\ldots(r-1)$ and thence the coefficient of e^r in $\log(1-e\cos u)$ is

$$-\left\{\frac{1}{r}\cos^r m-\frac{1}{1}\cos^{r-2}m\sin^2 m-\frac{1}{1.2}\frac{d}{dm}(\cos^{r-3}m\sin^3 m)-\&c.\right\},$$

continued as long as the exponent of $\cos m$ is not negative. Now in the expansion of $\frac{1}{\alpha}\log(1-\alpha e\cos\phi)$ where $\phi-\frac{1}{\alpha}\tan\phi=m$ the coefficient of e^r is $-\frac{1}{r}\alpha^{r-1}\cos^r\phi$ which (by Lagrange's theorem) is equal to

$$-\frac{1}{r}\alpha^{r-1}\left\{\cos^r m-\frac{1}{1.\alpha}r\cos^{r-1}m\sin m\tan m\right.$$
$$\left.-\frac{1}{1.2\alpha^2}\frac{d}{dm}(r\cos^{r-1}m\sin m\tan^2 m)-\&c.\right\}$$

$$=-\left\{\frac{1}{r}\alpha^{r-1}\cos^r m-\frac{1}{1}\alpha^{r-2}\cos^{r-2}m\sin^2 m\right.$$
$$\left.-\frac{1}{1.2}\alpha^{r-3}\cos^{r-3}m\sin^3 m-\&c.\right\},$$

where the series is continued indefinitely, but if we reject the negative powers of α and then put α equal to unity this is precisely equal to the former expression for the coefficient of e^r and the formula is thus shown to be true.

2, Stone Buildings, W.C.,
17th Nov., 1857.

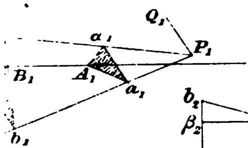
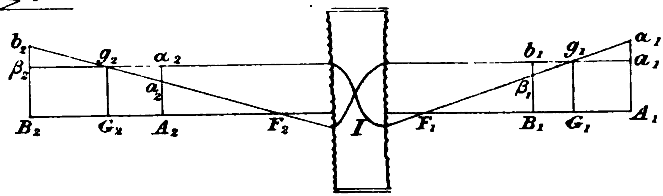


Fig. 2.



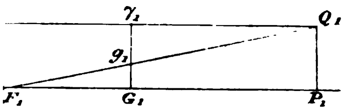
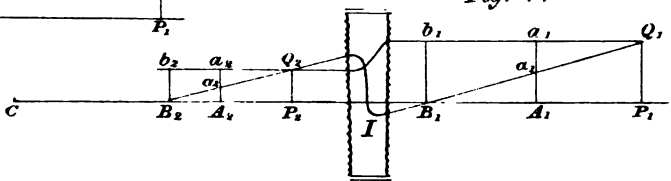


Fig. 4.



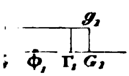
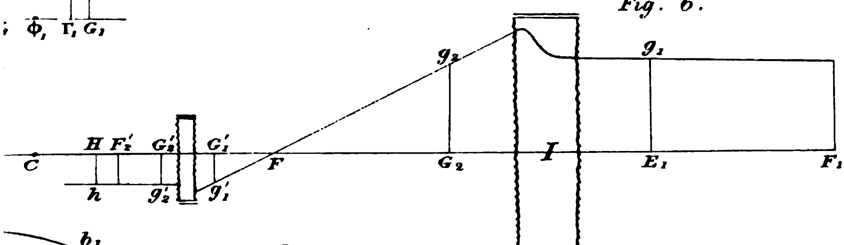


Fig. 6.



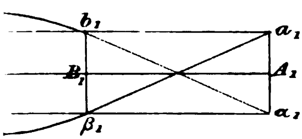
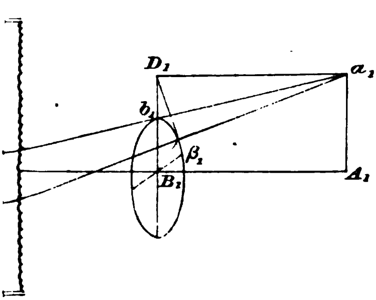
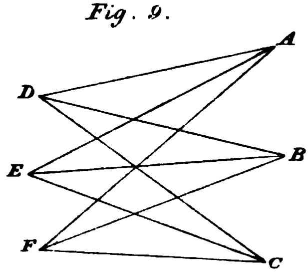


Fig. 9.





W. Metcalfe, Lith.

ON THE GENERAL LAWS OF OPTICAL INSTRUMENTS.

By J. C. Maxwell.

THE optical effects of compound instruments have been generally deduced from those of the elementary parts of which they are composed. The formulæ given in most works on Optics for calculating the effect of each spherical surface are simple enough, but, when we attempt to carry on our calculations from one of these surfaces to the next, we arrive at fractional expressions so complicated as to make the subsequent steps very troublesome.

Euler (Acad. R. de Berlin, 1757, 1761. Acad. R. de Paris, 1765) has attacked these expressions, but his investigations are not easy reading. Lagrange (Acad. Berlin, 1778, 1803) has reduced the case to the theory of continued fractions and so obtained general laws.

Gauss (*Dioptrische Untersuchungen*, Göttingen, 1841) has treated the subject with that combination of analytical skill with practical ability which he displays elsewhere, and has made use of the properties of principal foci and principal planes. An account of these researches is given by Prof. Miller in the third volume of Taylor's *Scientific Memoirs*. It is also given entire in French by M. Bravais in *Liouville's Journal* for 1856, with additions by the translator.

The method of Gauss has been followed by Prof. Listing in his *Treatise on the Dioptrics of the Eye* (in Wagner's *Handwörterbuch der Physiologie*) from whom I copy these references, and by Prof. Helmholtz in his *Treatise on Physiological Optics* (in Karsten's *Cyclopädie*).

The earliest general investigations are those of Cotes, given in Smith's *Optics*, II. 76 (1738). The method there is geometrical, and perfectly general, but proceeding from the elementary cases to the more complex by the method of mathematical induction. Some of his modes of expression, as for instance his measure of "apparent distance," have never come into use, although his results may easily be expressed more intelligibly; and indeed the whole fabric of Geometrical Optics, as conceived by Cotes and laboured by Smith, has fallen into neglect, except among the writers before named. Smith tells us that it was with reference to these optical theorems that Newton said "If Mr. Cotes had lived we might have known something."

The investigations which I now offer are intended to show how simple and how general the theory of instruments may be rendered, by considering the optical effects of the entire instrument, without examining the mechanism by which those effects are obtained. I have thus established a theory of "perfect instruments," geometrically complete in itself, although I have also shown, that no instrument depending on refraction and reflexion, (except the plane mirror) can be optically perfect. The first part of this theory was communicated to the Philosophical Society of Cambridge, 28th April, 1856, and an abstract will be found in the *Philosophical Magazine*, November, 1856. Propositions VIII. and IX. are now added. I am not aware that the last has been proved before.

In the following propositions I propose to establish certain rules for determining, from simple data, the path of a ray of light after passing through any optical instrument, the position of the conjugate focus of a luminous point, and the magnitude of the image of a given object. The method which I shall use does not require a knowledge of the internal construction of the instrument and derives all its data from two simple experiments.

There are certain defects incident to optical instruments from which, in the elementary theory, we suppose them to be free. A perfect instrument must fulfil three conditions:

I. Every ray of the pencil, proceeding from a single point of the object, must, after passing through the instrument, converge to, or diverge from, a single point of the image. The corresponding defect, when the emergent rays have not a common focus, has been appropriately called (by Dr. Whewell) *Astigmatism.*

II. If the object is a plane surface, perpendicular to the axis of the instrument, the image of any point of it must also lie in a plane perpendicular to the axis. When the points of the image lie in a curved surface, it is said to have the defect of *curvature.*

III. The image of an object on this plane must be similar to the object, whether its linear dimensions be altered or not; when the image is not similar to the object, it is said to be *distorted.*

An image free from these three defects is said to be *perfect.*

In the figure (fig. 1) let $A_1a_1a_1$ represent a plane object perpendicular to the axis of an instrument represented by I., then if the instrument is perfect, as regards an object at that distance, an image $A_2a_2\alpha_2$ will be formed by the emergent rays, which will have the following properties:

I. Every ray, which passes through a point a_1 of the object, will pass through the corresponding point a_2 of the image.

II. Every point of the image will lie in a plane perpendicular to the axis.

III. The figure $A_2a_2\alpha_2$ will be similar and similarly situated to the figure $A_1a_1\alpha_1$.

Now let us assume that the instrument is also perfect as regards an object in the plane $B_1b_1\beta_1$ perpendicular to the axis through B_1, and that the image of such an object is in the plane $B_2b_2\beta_2$ and similar to the object, and we shall be able to prove the following proposition:

PROP. I. If an instrument give a perfect image of a plane object at two different distances from the instrument, *all* incident rays having a common focus will have a common focus after emergence.

Let P_1 be the focus of incident rays. Let $P_1a_1b_1$ be any incident ray. Then, since every ray which passes through a_1 passes through a_2, its image after emergence, and since every ray which passes through b_1 passes through b_2, the direction of the ray $P_1a_1b_1$ after emergence must be a_2b_2.

Similarly, since α_2 and β_2 are the images of α_1 and β_1, if $P_1\alpha_1\beta_1$ be any other ray, its direction after emergence will be $\alpha_2\beta_2$.

Join $a_1\alpha_1$, $b_1\beta_1$, $a_2\alpha_2$, $b_2\beta_2$; then, since the parallel planes $A_1a_1\alpha_1$ and $B_1b_1\beta_1$ are cut by the plane of the two rays through P_1, the intersections $a_1\alpha_1$ and $b_1\beta_1$ are parallel.

Also, their images, being similarly situated, are parallel to them, therefore $a_2\alpha_2$ is parallel to $b_2\beta_2$, and the lines a_2b_2 and $\alpha_2\beta_2$ are in the same plane, and therefore either meet in a point P_2 or are parallel.

Now take a third ray through P_1, not in the plane of the two former. After emergence it must either cut both, or be parallel to them. If it cuts both it must pass through the point P_2, and then every other ray must pass through P_2, for no line can intersect three lines, not in one plane, without passing through their point of intersection. If not, then all the emergent rays are parallel, which is a particular case of a perfect pencil. So that for every position of the focus of incident rays, the emergent pencil is free from *astigmatism.*

PROP. II. In an instrument, perfect at two different distances, the image of any plane object perpendicular to the axis will be free from the defects of curvature and distortion.

Through the point P_1 of the object draw any line P_1Q_1 in the plane of the object, and through P_1Q_1 draw a plane cutting the planes A_1B_1 in the lines $a_1\alpha_1$, $b_1\beta_1$. These lines will be parallel to P_1Q_1 and to each other, wherefore also their images, $a_2\alpha_2$, $b_2\beta_2$, will be parallel to P_1Q_1 and to each other, and therefore in one plane.

Now suppose another plane drawn through P_1Q_1 cutting the planes A_1 and B_1 in two other lines parallel to P_1Q_1. These will have parallel images in the planes A_2 and B_2, and the intersection of the planes passing through the two pairs of images will define the line P_2Q_2 which will be parallel to them, and therefore to P_1Q_1, and will be the *image* of P_1Q_1. Therefore P_2Q_2, the image of P_1Q_1 is parallel to it, and therefore in a plane perpendicular to the axis. Now if all corresponding lines in any two figures be parallel, however the lines be drawn, the figures are similar, and similarly situated.

From these two propositions it follows that an instrument giving a perfect image at two different distances will give a perfect image at all distances. We have now only to determine the simplest method of finding the position and magnitude of the image, remembering that wherever two rays of a pencil intersect, all other rays of the pencil must meet, and that all parts of a plane object have their images in the same plane, and equally magnified or diminished.

PROP. III. A ray is incident on a perfect instrument parallel to the axis, to find its direction after emergence.

Let a_1b_1 (fig. 2) be the incident ray, A_1a_1 one of the planes at which an object has been ascertained to have a perfect image. A_2a_2 that image, similar to A_1a_1 but in magnitude such that $A_2a_2 = xA_1a_1$.

Similarly let B_2b_2 be the image of B_1b_1, and let $B_2b_2 = yB_1b_1$. Also let $A_1B_1 = c_1$ and $A_2B_2 = c_2$.

Then since a_2 and b_2 are the images of a_1 and b_1, the line $F_2a_2b_2$ will be the direction of the ray after emergence, cutting the axis in F_2, (unless $x=y$, when a_2b_2 becomes parallel to the axis). The point F_2 may be found, by remembering that $A_1a_1 = B_1b_1$, $A_2a_2 = xA_1a_1$, $B_2b_2 = yB_1b_1$. We find—

$$A_2F_2 = c_2\frac{x}{y-x}.$$

Let g_2 be the point at which the emergent ray is at the same distance from the axis as the incident ray, draw g_2c_2 perpendicular to the axis, then we have

$$F_2G_2 = \frac{c_2}{y-x}.$$

Similarly, if $\alpha_1\beta_1F_1$ be a ray, which, after emergence, becomes parallel to the axis; and G_1g_1 a line perpendicular to the axis, equal to the distance of the parallel emergent ray, then

$$A_1F_1 = c_1\frac{y}{x-y}, \quad F_1G_1 = \frac{c_1xy}{x-y}.$$

Definitions.

I. The point F_1, the focus of incident rays when the emergent rays are parallel to the axis, is called the *first principal focus* of the instrument.

II. The plane G_1g_1 at which incident rays through F_1 are at the same distance from the axis as they are after emergence, is called the *first principal plane* of the instrument. F_1G_1 is called the *first focal length.*

III. The point F_2, the focus of emergent rays when the incident rays are parallel, is called the *second principal focus.*

IV. The plane G_2g_2, at which the emergent rays are at the same distance from the axis, as before incidence, is called the *second principal plane*, and F_2G_2 is called the *second focal length.*

When $x=y$, the ray is parallel to the axis, both at incidence and emergence, and there are no such points as F and G. The instrument is then called a *telescope.* $x\,(=y)$ is called the *linear magnifying power* and is denoted by l, and the ratio $\frac{c_2}{c_1}$ is denoted by n, and may be called the *elongation.*

In the more general case, in which x and y are different, the principal foci and principal planes afford the readiest means of finding the position of images.

PROP. IV. Given the principal foci and principal planes of an instrument, to find the relations of the foci of the incident and emergent pencils.

Let F_1F_2 (fig. 3) be the principal foci, G_1G_2 the principal planes, Q_1 the focus of incident light, Q_1P_1 perpendicular to the axis.

Through Q_1 draw the ray $Q_1g_1F_1$. Since this ray passes through F_1 it emerges parallel to the axis, and at a distance from it equal to G_1g_1. Its direction after emergence is therefore Q_2g_2 where $G_2g_2 = G_1g_1$. Through Q_1 draw $Q_1\gamma_1$ parallel to the axis. The corresponding emergent ray will pass through F_2, and will cut the second principal plane at a distance $G_2\gamma_2 = G_1\gamma_1$, so that $F_2\gamma_2$ is the direction of this ray after emergence.

Since both rays pass through the focus of the emergent pencil, Q_2, the point of intersection, is that focus. Draw Q_2P_2 perpendicular to the axis. Then $P_1Q_1 = G_1\gamma_1 = G_2\gamma_2$, and $G_1g_1 = G_2g_2 = P_2Q_2$. By similar triangles $F_1P_1Q_1$ and $F_1G_1g_1$

$$P_1F_1 : F_1G_1 :: P_1Q_1 : (G_1g_1 =) \, P_2Q_2.$$

And by similar triangles $F_2P_2Q_2$ and $F_2G_2\gamma_2$

$$P_1Q_1 \, (= G_2\gamma_2) : P_2Q_2 :: G_2F_2 : F_2P_2.$$

We may put these relations into the concise form

$$\frac{P_1F_1}{F_1G_1} = \frac{P_1Q_1}{P_2Q_2} = \frac{G_2F_2}{F_2P_2},$$

and the values of F_2P_2 and P_2Q_2 are

$$F_2P_2 = \frac{F_1G_1 . G_2F_2}{P_1F_1} \text{ and } P_2Q_2 = \frac{F_1G_1}{P_1F_1} P_1Q_1.$$

These expressions give the distance of the image from F_2 measured along the axis, and also the perpendicular distance from the axis, so that they serve to determine completely the position of the image of any point, when the principal foci and principal planes are known.

PROP. V. To find the focus of emergent rays, when the instrument is a *telescope*.

Let Q_1 (fig. 4) be the focus of incident rays, and let $Q_1a_1b_1$ be a ray parallel to the axis; then, since the instrument is telescopic, the emergent ray $Q_2a_2b_2$ will be parallel to the axis, and $Q_2P_2 = l . Q_1P_1$.

Let $Q_1\alpha_1B_1$ be a ray through B_1, the emergent ray will be $Q_2\alpha_2B_2$, and $A_2\alpha_2 = l . A_1\alpha_1$.

$$\text{Now} \quad \frac{P_2B_2}{A_2B_2} = \frac{P_2Q_2}{A_2\alpha_2} = \frac{l . P_1Q_1}{l . A_1\alpha_1} = \frac{P_1Q_1}{A_1\alpha_1} = \frac{P_1B_1}{A_1B_1},$$

$$\text{so that} \quad \frac{P_2B_2}{P_1B_1} = \frac{A_2B_2}{A_1B_1} = n, \text{ a constant ratio.}$$

Cor. If a point C be taken on the axis of the instrument so that

$$CB_2 = \frac{A_2B_2}{A_1B_1 - A_2B_2}, \quad B_1B_2 = \frac{n}{1-n}\,B_1B_2,$$

then $$CP_2 = n.CP_1.$$

Def. The point C is called the *centre* of the telescope.

It appears, therefore, that the image of an object in a telescope has its dimensions perpendicular to the axis equal to l times the corresponding dimensions of the object, and the distance of any part from the plane through C equal to n times the distance of the corresponding part of the object. Of course all longitudinal distances among objects must be multiplied by n to obtain those of their images, and the tangent of the angular magnitude of an object as seen from a given point in the axis must be multiplied by $\frac{l}{n}$ to obtain that of the image of the object as seen from the image of the given point. The quantity $\frac{l}{n}$ is therefore called the *angular magnifying power*, and is denoted by m.

PROP. VI. To find the principal foci and principal planes of a combination of two instruments having a common axis.

Let I, I' (fig. 5) be the two instruments, $G_1F_1F_2G_2$ the principal foci and planes of the first, $G_1'F_1'F_2'G_2'$ those of the second, $\Gamma_1\phi_1\phi_2\Gamma_2$ those of the combination. Let the ray $g_1g_2g_1'g_2'$ pass through both instruments, and let it be parallel to the axis before entering the first instrument. It will therefore pass through F_2 the second principal focus of the first instrument, and through g_2 so that $G_2g_2 = G_1g_1$.

On emergence from the second instrument it will pass through ϕ_2 the focus conjugate to F_2, and through g_2' in the second principal plane, so that $G_2'g_2' = G_1'g_1'$. ϕ_2 is by definition the second principal focus of the combination of instruments, and if $\Gamma_2\gamma_2$ be the second principal plane, then $\Gamma_2\gamma_2 = G_1g_1$.

We have now to find the positions of ϕ_2 and Γ_2.

By Prop. IV., we have

$$F_2'\phi_2 = \frac{F_1'G_1'.G_2'F_2'}{F_2F_1'}.$$

Or, the distance of the principal focus of the combination, from that of the second instrument, is equal to the product of the focal lengths of the second instrument, divided by the distance of the second principal focus of the first instrument from the first of the second. From this we get

$$G_2'F_2' - F_2'\phi_2 = \frac{G_2'F_2'(F_2F_1' - F_1'G_1')}{F_2F_1'},$$

or

$$G_2'\phi_2 = \frac{G_2'F_2'.G_1'F_2}{F_2F_1'}.$$

Now, by the pairs of similar triangles $\phi G_2'g_2'$, $\phi\Gamma_2\gamma_2$ and $F_2G_1'g_1'$, $F_2G_2g_2$,

$$\frac{\Gamma_2\phi_2}{G_2'\phi_2} = \frac{\Gamma_2\gamma_2}{G_2'g_2} = \frac{G_2g_2}{G_1'g_1'} = \frac{F_2G_2}{G_1'F_2}.$$

Multiplying the two sides of the former equation respectively by the first and last of these equal quantities, we get

$$\Gamma_2\phi_2 = \frac{G_2F_2.G_2'F_2'}{F_2F_1'}.$$

Or, the second focal distance of a combination is the product of the second focal lengths of its two components, divided by the distance of their consecutive principal foci.

If we call the focal distances of the first instrument f_1 and f_2, those of the second f_1' and f_2', and those of the combination $\bar{f}_1$, $\bar{f}_2$, and put $F_2F_1' = d$, then the positions of the principal foci are found from the values

$$\phi_1F_1 = \frac{f_1f_2}{d}, \quad F_2'\phi_2 = \frac{f_1'f_2'}{d},$$

and the focal lengths of the combination from

$$\bar{f}_1 = \frac{f_1f_1'}{d}, \quad \bar{f}_2 = \frac{f_2f_2'}{d}.$$

When $d = 0$, all these values become infinite, and the compound instrument becomes a telescope.

PROP. VII. To find the linear magnifying power, the elongation, and the centre of the instrument, when the combination becomes a telescope.

Here (fig. 6) the second principal focus of the first instrument coincides at F with the first of the second. (In the figure, the focal distances of both instruments are taken in the opposite direction from that formerly assured. They are therefore to be regarded as *negative*).

In the first place, F_2' is conjugate to F_1, for a pencil whose focus before incidence is F_1 will be parallel to the axis between the instruments, and will converge to F_2' after emergence.

Also if G_1g_1 be an object in the first principal plane, G_2g_2 will be its first image, equal to itself, and if Hh be its final image

$$F_2'H = \frac{FG_1'.G_2'F_2'}{G_2F} = \frac{f_1'f_2'}{f_2},$$

$$Hh = \frac{FG_1'}{G_2F}G_2g_2 = -\frac{f_1'}{f_2}G_1g_1.$$

Now the linear magnifying power is $\frac{Hh}{G_1g_1}$, and the elongation is $\frac{F_2'H}{F_1G_1}$, because F_2' and H are the images of F_1 and G_1 respectively; therefore

$$l = -\frac{f_1'}{f_2}, \quad \text{and } n = \frac{f_1'f_2'}{f_1f_2}.$$

The angular magnifying power $= m = \frac{l}{n} = -\frac{f_1}{f_2'}$.

The centre of the telescope is at the point C, such that

$$F_2'C = \frac{n}{1-n}F_1F_2'.$$

When n becomes 1 the telescope has no centre. The effect of the instrument is then simply to alter the position of an object by a certain distance measured along the axis, as in the case of refraction through a plate of glass bounded by parallel planes. In certain cases this constant distance itself disappears, as in the case of a combination of three convex lenses of which the focal lengths are 4, 1, 4 and the distances 4 and 4. This combination simply inverts every object without altering its magnitude or distance along the axis.

The preceding theory of perfect instruments is quite independent of the mode in which the course of the rays is changed within the instrument, as we are supposed to know only that the path of every ray is straight before it enters, and after it emerges from the instrument. We have now to consider, how far these results can be applied to actual instruments, in which the course of the rays is changed by reflexion or refraction. We know that such instruments may be made so as to fulfil approximately the conditions of a perfect instrument, but that absolute perfection has not yet been obtained. Let us inquire whether any additional general law of optical instruments can be deduced from the laws of reflexion and refraction, and whether the imperfection of instruments is necessary or removeable.

The following theorem is a necessary consequence of the known laws of reflexion and refraction, whatever theory we adopt.

If we multiply the length of the parts of a ray which are in different media by the indices of refraction of those media, and call the sum of these products the *reduced path* of the ray, then:

I. The extremities of all rays from a given origin, which have the same reduced path, lie in a surface normal to those rays.

II. When a pencil of rays is brought to a focus, the reduced path from the origin to the focus is the same for every ray of the pencil.

In the undulatory theory, the "reduced path" of a ray is the distance through which light would travel in space, during the time which the ray takes to traverse the various media, and the surface of equal "reduced paths" is the wave-surface. In *extraordinary* refraction the wave-surface is not always normal to the ray, but the other parts of the proposition are true in this and all other cases.

From this general theorem in optics we may deduce the following propositions, true for all instruments depending on refraction and reflexion.

PROP. VIII. In any optical instrument depending on refraction or reflexion, if $a_1\alpha_1$, $b_1\beta_1$ (fig. 7) be two objects and $a_2\alpha_2$, $b_2\beta_2$ their images, A_1B_1 the distance of the objects, A_2B_2 that of the images, μ_1 the index of refraction of the medium in which the objects are, μ_2 that of the medium in which the images are, then

$$\mu_1 \frac{a_1\alpha_1 \times b_1\beta_1}{A_1B_1} = \mu_2 \frac{a_2\alpha_2 \times b_2\beta_2}{A_2B_2},$$

approximately, when the objects are small.

Since a_2 is the image of a_1, the reduced path of the ray $a_1b_1a_2$ will be equal to that of $a_1\beta_1a_2$, and the reduced paths of the rays $\alpha_1\beta_1\alpha_2$ and $\alpha_1b_1\alpha_2$ will be equal.

Also because $b_1\beta_1$ and $b_2\beta_2$ are conjugate foci, the reduced paths of the rays $b_1a_2b_2$ and $b_1a_2\beta_2$, and of $\beta_1a_2\beta_2$ and $\beta_1\alpha_2\beta_2$ will be equal. So that the reduced paths

$$\begin{aligned} a_1b_1 + b_1a_2 &= a_1\beta_1 + \beta_1a_2 \\ \alpha_1\beta_1 + \beta_1\alpha_2 &= \alpha_1b_1 + b_1\alpha_2 \\ b_1\alpha_2 + \alpha_2b_2 &= b_1a_2 + a_2\beta_2 \\ \beta_1a_2 + a_2\beta_2 &= \beta_1\alpha_2 + \alpha_2\beta_2 \end{aligned}$$

$$\therefore\ a_1b_1 + \alpha_1\beta_1 + \alpha_2b_2 + a_2\beta_2 = a_1\beta_1 + \alpha_1b_1 + a_2b_2 + \alpha_2\beta_2,$$

these being still the *reduced paths* of the rays, that is, the length of each ray multiplied by the index of refraction of the medium.

If the figure is symmetrical about the axis, we may write the equation

$$\mu_1(a_1\beta_1 - a_1b_1) = \mu_2(a_2\beta_2 - a_2b_2),$$

where $a_1\beta_1$, &c. are now the *actual lengths* of the rays so named.

Now $$\overline{a_1\beta_1}^2 = \overline{A_1B_1}^2 + \tfrac{1}{4}(a_1\alpha_1 + b_1\beta_1)^2,$$

$$\overline{a_1b_1}^2 = \overline{A_1B_1}^2 + \tfrac{1}{4}(a_1\alpha_1 - b_1\beta_1)^2,$$

so that $$\overline{a_1\beta_1}^2 - \overline{a_1b_1}^2 = a_1\alpha_1 \times b_1\beta_1,$$

and $$\mu_1(a_1\beta_1 - a_1b_1) = \mu_1\frac{a_1\alpha_1 \times b_1\beta_1}{a_1\beta_1 + a_1b_1}.$$

Similarly $$\mu_2(a_2\beta_2 - a_2b_2) = \mu_2\frac{a_2\alpha_2 \times b_2\beta_2}{a_2\beta_2 + a_1b_1}.$$

So that the equation

$$\mu_1\frac{a_1\alpha_1 \times b_1\beta_1}{a_1\beta_1 + a_1b_1} = \mu_2\frac{a_2\alpha_2 \times b_2\beta_2}{a_2\beta_2 + a_2b_2}$$

is true accurately, and since when the objects are small, the denominators are nearly $2A_1B_1$ and $2A_2B_2$, the proposition is proved approximately true.

Using the expressions of Prop. III., this equation becomes

$$\mu_1\frac{1}{c_1} = \mu_2\frac{xy}{c_2}.$$

Now by Prop. III., when x and y are different, the focal lengths f_1 and f_2 are

$$f_1 = c_1\frac{xy}{x-y},\quad f_2 = c_2\frac{1}{y-x};$$

therefore $\dfrac{f_1}{f_2} = \dfrac{c_1 xy}{c_2} = \dfrac{\mu_1}{\mu_2}$ by the present theorem.

So that in any instrument, not a telescope, the focal lengths are directly as the indices of refraction of the media to which they belong. If, as in most cases, these media are the same, then the two focal distances are *equal.*

When $x=y$, the instrument becomes a telescope, and we have, by Prop. V., $l=x$, and $n=\dfrac{c_2}{c_1}$; and therefore by this theorem

$$\frac{\mu_1}{\mu_2} = \frac{l^2}{n}.$$

We may find l experimentally by measuring the actual diameter of the image of a known near object, such as the aperture of the object glass. If O be the diameter of the aperture and o that of the circle of light at the eye-hole (which is its image), then

$$l = \frac{o}{O}.$$

From this we find the elongation and the angular magnifying power

$$n = \frac{\mu_2}{\mu_1} l^2, \quad \text{and } m = \frac{\mu_1}{\mu_2}\frac{1}{l}.$$

When $\mu_1 = \mu_2$, as in ordinary cases, $m = \dfrac{1}{l} = \dfrac{O}{o}$, which is Gauss' rule for determining the magnifying power of a telescope.

Prop. IX. It is impossible, by means of any combination of reflexions and refractions, to produce a *perfect* image of an object at two different distances, unless the instrument be a telescope, and

$$l = n = \frac{\mu_2}{\mu_1}, \quad m = 1.$$

It appears from the investigation of Prop. VIII. that the results there obtained, if true when the objects are very small, will be incorrect when the objects are large, unless

$$a_1\beta_1 + a_1b_1 : a_2\beta_2 + a_2b_2 :: A_1B_1 : A_2B_2,$$

and it is easy to prove that this cannot be, unless all the lines in the one figure are proportional to the corresponding lines in the other.

In this way we might show that we cannot in general have an astigmatic, plane, undistorted image of a plane object. But we can prove that we cannot get perfectly focussed images of an object in two positions, even at the expense of curvature and distortion.

We shall first prove that if two objects have perfect images, the reduced path of the ray joining any given points of the two objects is equal to that of the ray joining the corresponding points of the images.

Let a_2 (fig. 8) be the perfect image of a_1 and β_2 of β_1. Let $A_1a_1 = a_1$, $B_1\beta_1 = b_1$, $A_2a_2 = a_2$, $B_2\beta_2 = b_2$, $A_1B_1 = c_1$, $A_2B_2 = c_2$. Draw a_1D_1 parallel to the axis to meet the plane B_1, and a_2D_2 to the plane of B_2.

Since everything is symmetrical about the axis of the instrument we shall have the angles $D_1B_1\beta_1 = D_2B_2\beta_2 = \theta$, then in either figure, omitting the suffixes,

$$\overline{a\beta}^2 = \overline{aD}^2 + \overline{D\beta}^2$$
$$= c^2 + a^2 + b^2 - 2ab\cos\theta.$$

It has been shown in Prop. VIII. that the difference of the reduced paths of the rays a_1b_1, $a_1\beta_1$ in the object must be equal to the difference of the reduced paths of a_2b_2, $a_2\beta_2$ in the image. Therefore, since we may assume any value for θ,

$$\mu_1\sqrt{(a_1^2 + b_1^2 + c_1^2 - 2a_1b_1\cos\theta)} - \mu_2\sqrt{(a_2^2 + b_2^2 + c_2^2 - 2a_2b_2\cos\theta)}$$

is constant for all values of θ. This can be only when

$$\mu_1\sqrt{(a_1^2 + b_1^2 + c_1^2)} = \mu_2\sqrt{(a_2^2 + b_2^2 + c_2^2)},$$

and $$\mu_1\sqrt{(a_1b_1)} = \mu_2\sqrt{(a_2b_2)},$$

which shows that the constant must vanish, and that the lengths of lines joining corresponding points of the objects and of the images must be inversely as the indices of refraction before incidence and after emergence.

Next let ABC, DEF (fig. 9) represent three points in the one object and three points in the other object, the figure being drawn to a scale so that all the lines in the figure are the actual lines multiplied by μ_1. The lines of the figure represent the reduced paths of the rays between the corresponding points of the objects.

Now it may be shown that the form of this figure cannot be altered without altering the length of one or more of the nine lines joining the points ABC to DEF. Therefore since the reduced paths of the rays in the image are equal to those in the object, the figure must represent the image on a scale of μ_2 to 1, and therefore the instrument must magnify every part of the object alike and elongate the distances parallel to the axis in the same proportion. It is therefore a telescope, and $m=1$.

If $\mu_1=\mu_2$, the image is exactly equal to the object, which is the case in reflexion in a plane mirror, which we know to be a perfect instrument for all distances.

The only case in which by refraction at a single surface we can get a perfect image of more than one point of the object, is when the refracting surface is a sphere, radius r, index μ, and when the two objects are spherical surfaces, concentric with the sphere, their radii being $\frac{r}{\mu}$, and r; and the two images also concentric spheres, radii μr, and r.

In this latter case the image is perfect, only at these particular distances, and not generally.

I am not aware of any other case in which a perfect image of an object can be formed, the rays being straight before they enter, and after they emerge from the instrument. The only case in which perfect astigmatism for all pencils has hitherto been proved to exist, was suggested to me by the consideration of the structure of the crystalline lens in fish, and was published in one of the problem-papers of the *Cambridge and Dublin Mathematical Journal*. My own method of treating that problem is to be found in that *Journal*, for February, 1854. The case is that of a medium whose index of refraction varies with the distance from a centre, so that if μ_0 be its value at the centre, a a given line, and r the distance of any point where the index is μ, then

$$\mu = \mu_0 \frac{a^2}{a^2 + r^2}.$$

The path of every ray within this medium is a circle in a plane passing through the centre of the medium.

Every ray from a point in the medium, distant b from the centre, will converge to a point on the opposite side of the centre and distant from it $\frac{a^2}{b}$.

It will be observed that both the object and the image are included in the variable medium, otherwise the images would not be perfect. This case therefore forms no exception to the result of Prop. IX., in which the object and image are supposed to be outside the instrument.

Aberdeen, 12*th Jan*, 1858.

ON THE AREA OF THE CONIC SECTION, REPRESENTED BY THE GENERAL TRILINEAR EQUATION OF THE SECOND DEGREE.

By N. M. FERRERS.

SUPPOSE that the position of a point P is determined by reference to a triangle ABC; its coordinates x, y, z denoting respectively the ratios of the areas of the triangles PBC, PCA, PAB to the triangle ABC, so that x, y, z are connected by the identical relation

$$x+y+z=1.$$

Our object is to find the area of the conic section represented by the general equation

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=0.$$

If the conic section be orthogonally projected, the values of the coefficients, the values of x, y, z, and the ratio of the area of the conic to that of the triangle of reference, will all remain unaltered. Hence, if Δ denote the area of the triangle, S that of the conic section, $\frac{S}{\Delta}$ will be a function of the coefficients only. To find the form of this function, we observe (1) that it must be of 0 dimensions, (2) that it must vanish when the relation among the coefficients is such that the conic section degenerates into two straight lines, (3) that it will be infinite when the conic section is a parabola, *i.e.* touches the line $x+y+z=0$.

Hence, S vanishes when

$$AA'^2+BB'^2+CC'^2-ABC-2A'B'C'=0,$$

and is infinite when

$$A'^2-BC+B'^2-CA+C'^2-AB$$
$$+2(B'C'-AA')+2(C'A'-BB')+2(A'B'-CC')=0;$$

whence $\frac{S}{\Delta}$, being of 0 dimensions, must be some function of

$$\frac{(AA'^2+BB'^2+CC'^2-ABC-2A'B'C')^2}{\{A'^2-BC+B'^2-CA+C'^2-AB+2(B'C'-AA')+2(C'A'-BB')+2(A'B'-CC')\}^3}.$$

Representing this fraction for the present by V, suppose $\frac{S}{\Delta}=\phi(V)$.

To find the form of ϕ, suppose the conic section to be the circumscribed circle, and the sides of the triangle of reference a, b, c; then,

$$A = B = C = 0,$$

$$\frac{A'}{a^2} = \frac{B'}{b^2} = \frac{C'}{c^2},$$

whence
$$V = \frac{4a^4b^4c^4}{(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)^3},$$

also it is known that

$$\frac{S}{\Delta} = \frac{4\pi a^2b^2c^2}{(a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2)^{\frac{3}{2}}}$$

$$= 2\pi\,(V)^{\frac{1}{2}}.$$

Hence,

$$S = \frac{2\pi\,(AA'^2 + BB'^2 + CC'^2 - ABC - 2A'B'C')\,\Delta}{\{A'^2 - BC + B'^2 - CA + C'^2 - AB + 2(B'C' - AA') + 2(C'A' - BB') + 2(A'B' - CC')\}^{\frac{3}{2}}}$$

the required expression.

A similar method may be applied to determine the area of the conic section, when it is defined by the relation between the distances of its several tangents from three given points.

DIRECT INVESTIGATION OF THE QUESTION DISCUSSED IN THE FOREGOING PAPER.

By A. Cayley.

The position of a point P being determined as in the foregoing paper, let α, β, γ denote in like manner the coordinates of a point O, we have

$$\alpha + \beta + \gamma = 1,$$

and consequently if ξ, η, ζ are the relative coordinates $x - \alpha$, $y - \beta$, $z - \gamma$, we have

$$\xi + \eta + \zeta = 0.$$

The expression for the distance of the two points O, P is readily obtained in terms of the relative coordinates, viz. calling this distance r, we have

$$r^2 = L\xi^2 + M\eta^2 + N\zeta^2,$$

where, if l, m, n are the sides of the triangle ABC, we have

$$L = \tfrac{1}{2}(m^2 + n^2 + l^2),$$
$$M = \tfrac{1}{2}(n^2 + l^2 - m^2),$$
$$N = \tfrac{1}{2}(l^2 + m^2 - n^2).$$

And it is to be remarked that these values give

$$MN + NL + LM = \tfrac{1}{4}(2m^2n^2 + 2n^2l^2 + 2l^2m^2 - l^4 - m^4 - n^4) = 4\Delta^2,$$

if Δ denote the area of the triangle ABC.

Consider now a conic

$$(a, b, c, f, g, h)(x, y, z)^2,$$

and suppose as usual that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ are the inverse coefficients and that K is the discriminant, suppose also for shortness

$$P = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(1, 1, 1)^2.$$

The coordinates of the centre being α, β, γ, we have

$$\alpha = \frac{1}{P}(\mathfrak{A}, \mathfrak{H}, \mathfrak{G})(1, 1, 1),$$

$$\beta = \frac{1}{P}(\mathfrak{H}, \mathfrak{B}, \mathfrak{F})(1, 1, 1),$$

$$\gamma = \frac{1}{P}(\mathfrak{G}, \mathfrak{F}, \mathfrak{C})(1, 1, 1),$$

and writing as before ξ, η, ζ for $x - \alpha$, $y - \beta$, $z - \gamma$, so that ξ, η, ζ are the coordinates of a point P of the conic, in relation to the centre, we have x, y, z respectively equal to $\xi + \alpha$, $\eta + \beta$, $\zeta + \gamma$, and the equation of the conic gives

$$(a, \ldots)(\xi + \alpha, \ \eta + \beta, \ \zeta + \gamma)^2 = 0,$$

which may be written

$$\begin{aligned} &(a, \ldots)(\xi, \eta, \zeta)^2 \\ +\, 2\,&(a, \ldots)(\alpha, \beta, \gamma)(\xi, \eta, \zeta) \\ +\, &(a, \ldots)(\alpha, \beta, \gamma)^2 = 0. \end{aligned}$$

Now observing the equations

$$(a, h, g)(\alpha, \beta, \gamma) = \frac{K}{P},$$

$$(h, b, f)(\alpha, \beta, \gamma) = \frac{K}{P},$$

$$(g, f, c)(\alpha, \beta, \gamma) = \frac{K}{P},$$

we have

$$(a, \ldots)(\alpha, \beta, \gamma)(\xi, \eta, \zeta) = \frac{K}{P}(\xi + \eta + \zeta) = 0,$$

$$(a, \ldots)(\alpha, \beta, \gamma)^2 \qquad = \frac{K}{P}(\alpha + \beta + \gamma) = \frac{K}{P},$$

and the equation of the conic gives therefore

$$(a, \ldots)(\xi, \eta, \zeta)^2 + \frac{K}{P} = 0,$$

and we have as before

$$\xi + \eta + \zeta = 0.$$

To find the axes we have only to make

$$r^2 = L\xi^2 + M\eta^2 + N\zeta^2,$$

a maximum or minimum, ξ, η, ζ varying subject to the preceding two conditions, this gives

$$(a, h, g)(\xi, \eta, \zeta) + \lambda L\xi + \mu = 0,$$
$$(h, b, f)(\xi, \eta, \zeta) + \lambda M\eta + \mu = 0,$$
$$(g, f, c)(\xi, \eta, \zeta) + \lambda N\zeta + \mu = 0,$$

and multiplying by ξ, η, ζ, adding and reducing

$$-\frac{K}{P} + \lambda r^2 = 0,$$

which gives

$$\lambda = \frac{K}{Pr^2}.$$

Substituting this value, and joining to the resulting three equations the equation

$$\xi + \eta + \zeta = 0.$$

We may eliminate ξ, η, ζ, μ, and the result is

$$\begin{vmatrix} a + \frac{KL}{Pr^2}, & h, & g, & 1 \\ h, & b + \frac{KM}{Pr^2}, & f, & 1 \\ g, & f, & c + \frac{KN}{Pr^2}, & 1 \\ 1, & 1, & 1 & \end{vmatrix} = 0,$$

which may also be written

$$(\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{F}', \mathfrak{G}', \mathfrak{H}')(1, 1, 1)^2 = 0,$$

where $(\mathfrak{A}', \ldots)$ are what $(\mathfrak{A}, \ldots)$ become when a, b, c are changed into $a + \frac{KL}{Pr^2}$, $b + \frac{KM}{Pr^2}$, $c + \frac{KN}{Pr^2}$, we in fact have

$$\mathfrak{A}' = \mathfrak{A} + \frac{K}{Pr^2}(bN + cM) + \frac{K^2}{P^2r^4} MN,$$
$$\vdots$$
$$\mathfrak{F}' = \mathfrak{F} - \frac{K}{Pr^2} Lf,$$
$$\vdots$$

and observing the value of P the result consequently is

$$P + \frac{K}{Pr^2}\{(b + c - 2f) L + (c + a - 2g) M + (a + b - 2h) N\} + \frac{K^2}{P^2r^4}(MN + NL + LM) = 0,$$

which may also be written

$$P^2r^4 + PKr^2\{(b+c-2f)L + (c+a-2g)M + (a+b-2h)N\} + 4\Delta^2K^2 = 0.$$

Hence if r_1, r_2 are the two semiaxes, we have

$$r_1^2 r_2^2 = \frac{4\Delta^2K^2}{P^3},$$

and the area is $\pi r_1 r_2$ which is equal to

$$\frac{2\pi K\Delta}{\sqrt{(P^3)}},$$

which agrees with Mr. Ferrers' result.

The formula $r^2 = L\xi^2 + M\eta^2 + N\zeta^2$ which is assumed in the preceding investigation may be proved as follows:

Writing a, b, c (instead of l, m, n) for the sides of the fundamental triangle and A, B, C for the angles, the equation in question is

$$r^2 = bc \cos A\, \xi^2 + ca \cos B\, \eta^2 + ab \cos C\, \zeta^2.$$

Now writing α, β, γ for the inclinations of the line r to the sides of the triangle, we have

$$A = \beta - \gamma,$$
$$B = \gamma - \alpha,$$
$$C = \pi + \alpha - \beta.$$

Moreover taking for a moment λ, μ, ν to denote the perpendiculars from the angles on the opposite sides, we have

$$\lambda = c\sin B = b\sin C,$$
$$\mu = a\sin C = c\sin A,$$
$$\nu = b\sin A = a\sin B.$$

And

$$\xi = \frac{r\sin\alpha}{\lambda},\quad \eta = \frac{r\sin\beta}{\mu},\quad \zeta = \frac{r\sin\gamma}{\nu},$$

the values of ξ^2, η^2, ζ^2 consequently are

$$\frac{r^2\sin^2\alpha}{bc\sin B\sin C},\quad \frac{r^2\sin^2\beta}{ca\sin C\sin A},\quad \frac{r^2\sin^2\gamma}{ab\sin A\sin B},$$

and the equation to be proved becomes

$$1 = \frac{\cos A\sin^2\alpha}{\sin B\sin C} + \frac{\cos B\sin^2\beta}{\sin C\sin A} + \frac{\cos C\sin^2\gamma}{\sin A\sin B},$$

or what is the same thing

$$\sin A\sin B\sin C$$
$$= \sin A\cos A\sin^2\alpha + \sin B\cos B\sin^2\beta + \sin C\cos C\sin^2\gamma,$$

or again

$$4\sin A\sin B\sin C$$
$$= \sin 2A(1-\cos 2\alpha) + \sin 2B(1-\cos 2\beta) + \sin 2C(1-\cos 2\gamma),$$

or putting for A, B, C their values in terms of α, β, γ this is

$$-4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta) = \sin(2\beta-2\gamma)(1-\cos 2\alpha)$$
$$+\sin(2\gamma-2\alpha)(1-\cos 2\beta)$$
$$+\sin(2\alpha-2\beta)(1-\cos 2\gamma),$$

which is an identical equation; it is most readily proved by writing x, y, z for $\tan\alpha$, $\tan\beta$, $\tan\gamma$; the equation thus becomes

$$\frac{-4}{(1+x^2)(1+y^2)(1+z^2)}(y-z)(z-x)(x-y)$$
$$= \Sigma\frac{1}{(1+y^2)(1+z^2)}\{2y(1-z^2)-2z(1-y^2)\}\frac{2x^2}{1+x^2},$$

or multiplying out

$$-(y-z)(z-x)(x-y) = \Sigma(y-z)(1+yz)x^2$$
$$= \Sigma x^2(y-z) + xyz\Sigma x(y-z),$$

that is

$$-(y-z)(z-x)(x-y)=\Sigma x^2(y-z)=x^2(y-z)+y^2(z-x)+z^2(x-y),$$

which is an identity.

[The value of r^2 may also be found as follows:

r^2 is a rational homogeneous function of ξ, η, ζ of the second degree; hence, since

$$\xi+\eta+\zeta=0,$$

and therefore

$$2\eta\zeta=\xi^2-\eta^2-\zeta^2, \quad 2\zeta\xi=\eta^2-\zeta^2-\xi^2, \quad 2\xi\eta=\zeta^2-\xi^2-y^2,$$

r^2 must be expressible under the form $L\xi^2+M\eta^2+N\zeta^2$; and it remains to determine L, M, N.

For this purpose, suppose the two points, whose distance is sought, to be B, C, then

$$\xi=0, \quad \eta=1, \quad \zeta=1, \quad \text{and } r=a.$$

Hence $$M+N=a^2.$$

Similarly $$N+L=b^2,$$

$$L+M=c^2;$$

therefore $$L=\frac{b^2+c^2-a^2}{2}, \quad M=\frac{c^2+a^2-b^2}{2}, \quad N=\frac{a^2+b^2-c^2}{2},$$

the same values as those already given.—N.M.F.]

ON THE FACTORIAL NOTATION.

By H. W. Elphinstone, M.A., Trinity College, Cambridge.

(Continued from p. 128.)

12. To expand $x^n|^1 \times x^b|^1$ into the sum of a series of factorials having x as their base.

Let

$$x^n|^1 x^b|^1 = x^{n+b}|^1 + A_b x^{n+b-1}|^1 + B_b x^{n+b-2}|^1 + \&c. + L_b x^{n+1}|^1 + M_b x^n|^1.$$

Then introducing the factor

$$\begin{aligned} x+b &= x+b+n-n \\ &= x+b+n-1-(n-1) \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \\ &= x+n-(n-b), \end{aligned}$$

we have

$$x^n|^1 x^{b+1}|^1 = x^{n+b+1}|^1 + \left.\begin{matrix} A_b \\ -n \end{matrix}\right\} x^{n+b}|^1 + \left.\begin{matrix} B_b \\ -(n-1)A_b \end{matrix}\right\} x^{n+b-1}|^1 + \ldots$$

$$+ \left.\begin{matrix} M_b \\ -(n-b+1) \end{matrix}\right\} x^{n+1}|^1 - (n-b) M_b x^n|^1,$$

whence we see that

$$A_{b+1} = A_b - n, \quad B_{b+1} = B_b - (n-1) A_b, \quad C_{b+1} = C_b - (n-2) B_b,$$

and so on, or forming a table

	A	B	C	D	E
$b=1$	$-n$	0	0	0	0
2	$-2n$	$+n^2|^{-1}$	0	0	0
3	$-3n$	$+3n^2|^{-1}$	$-n^3|^{-1}$	0	0
4	$-4n$	$+6n^2|^{-1}$	$-4n^3|^{-1}$	$+n^4|^{-1}$	0
5	$-5n$	$+10n^2|^{-1}$	$-10n^3|^{-1}$	$+5n^4|^{-1}$	$-n^5|^{-1}$

Now considering the numerical value only of the coefficients of these factorials and calling the order of the column u, we see that

$$(b, u) = (b-1)(u-1) + (b-1)u,$$

which is a property of the binomial coefficients. It follows that

$$x^{n|1}x^{b|1} = x^{n+b|1} - bnx^{n+b-1|1} + \frac{b^{2|-1}}{1^{2|1}}\frac{n^{2|-1}}{1^{2|1}}x^{n+b-2|1} - \frac{b^{3|-1}}{1^{3|1}}\frac{n^{3|-1}}{1^{3|1}}x^{n+b-3|1} + \&c.$$

With the following similar formulas $n > b$,

$$x^{n|-1}x^{b|-1} = x^{n+b|-1} + nbx^{n+b-1|-1} + \frac{n^{2|-1}b^{2|-1}}{1^{2|+1}1^{2|1}}x^{n+b-2|-1} + \&c. + x^{n|-1},$$

$$x^{n|1}\ x^{b|-1} = x^{n-b|1} - nbx^{n-b-1|1} + \&c.,$$

$$x^{n|-1}x^{b|1} = x^{n-b|-1} + nbx^{n-b-1|1} + \&c.$$

13. To determine the coefficients A, B, C... in the equation

$$(x+a_1)(x+a_2)\dots(x+a_n) = A_n x^{n|1} + B_n x^{n-1|1} + \dots + N_n,$$

where the subscribed indices n on the right-hand side shew that there are n factors on the left.

According to our notation

$$x + a_1 = A_1 x^{1|1} + B_1,$$

so that $$A_1 = 1, \quad B_1 = a_1,$$

Also

$$(x+a_1)\dots(x+a_{r-1}) = A_{r-1}x^{r-1|1} + B_{r-1}x^{r-2|1} + C_{r-1}x^{r-3|1} + \dots R_{r-1}.$$

Introducing the factor

$$x + r - 1 + a_r - r + 1 = x + r - 2 + a_r - r + 2 = \dots = x + a_r,$$

we have $(x+a_1)(x+a_2)\dots(x+a_r)$

$$= A_{r-1}x^{r-1|1}(x+r-1+a_r-r+1) + B_{r-1}x^{r-2|1}(x+r-2+a_r-r+2) + \&c.$$
$$+ \dots + R_{r-1}(x+a_r)$$

$$= A_{r-1}x^{r|1} + \left.\begin{matrix} B_{r-1} \\ +(a_r-r+1)A_{r-1}\end{matrix}\right\} x^{r-1|1} + \left.\begin{matrix} C_{r-1} \\ +(a_r-r+2)B_{r-1}\end{matrix}\right\} x^{r-2|1} + \dots + a_r R_{r-1}.$$

It follows that

$$A_r = A_{r-1}, \quad B_r = B_{r-1} + (a_r - r + 1)A_{r-1}, \quad C_r = C_{r-1} + (a_r - r + 2)B_{r-1}, \ \&c.,$$

ending with $$S_r = a_r R_{r-1},$$

on which we may remark that $A_r = 1$ for

$$A_r = A_{r-1} = A_{r-2} = \dots = A_1 = 1.$$

The other coefficients can be readily obtained by the following tables:

I	II	III	IV	V
a_1	a_2	a_3	a_4	a_5
	a_2-1	a_3-1	a_4-1	a_5-1
		a_3-2	a_4-2	a_5-2
			a_4-3	a_5-3
				a_5-4

I	*B*	II	*P*	*C*	III	*Q*	*D*
a_1	B_1	a_2	B_1a_2	C_2	a_3	C_2a_3	D_3
a_2-1	B_2	a_3-1	$(a_3-1)B_2$	C_3	a_4-1	$(a_4-1)C_3$	D_4
a_3-2	B_3	a_4-2	$(a_4-2)B_3$	C_4	a_5-2	$(a_5-2)C_4$	D_5
a_4-3	B_4	a_5-3	$(a_5-3)B_4$	C_5			
a_5-4	B_5						

et cætera.

The law of the first table is obvious. The second table is formed as follows:

Each of the vertical columns I, II, III, &c. is the same as the corresponding oblique column in the first table; a number in one of the columns *B*, *C*, *D*, &c. is the sum of the number in the same horizontal line and preceding column, and the number in the same column and preceding horizontal line: a number in either of the columns *P*, *Q*, *R*, &c. is the product of the numbers in the same horizontal line and two preceding columns.

As an example, let us transform

$$(x+2)(x-1)(x+7)$$

into such a form

I	II	III
2	−1	7
	−2	6
		+5

I	*B*	II	*P*	*C*	III	*Q*	*D*
2	2	−1	−2	−2	7	−14	−14
−2	0	6	0	−2			
5	5						

$$(x+2)(x-1)(x+7) = x^{3|1} + 5x^{2|1} - 2x^{1|1} - 14,$$

It is worth while taking particular notice of the case in which we wish to put x^n under the above mentioned form; for in this case the quantities $a_1, a_2, \ldots a_n$ are all equal to each other and to zero, so that the columns I, II, III, &c. are all identical, and we need write down the first one of them only.

If we have any expression

$$a_n x^n + a_{n-1} x^{n-1} + \&c. + a_0,$$

we can easily put it under the given form, by putting each of the quantities x^n, x^{n-1}, &c. under that form and taking their sum after multiplying them by the coefficients a_n, a_{n-1}, &c. a_0 respectively: and here we may observe that the operation of transforming x^n gives us the coefficients for the transformation of all lower powers of x.

Suppose for instance that we were required to find the finite integral of

$$24x^5 + 18x^4 - 3x^3 + 50x^2 - 20x + 7,$$

I	*B*	*P*	*C*	*Q*	*D*	*R*	*E*
0	0	0	0	0	0	0	0
−1	− 1	+ 1	+ 1	− 1	− 1	+1	1
−2	− 3	+ 6	+ 7	−14	−15		
−3	− 6	+18	+25				
−4	−10						

Hence
$$\begin{aligned}
x^5 &= x^{5}|^{1} - 10x^{4}|^{1} + 25x^{3}|^{1} - 15x^{2}|^{1} + x^{1}|^{1}\\
x^4 &= x^{4}|^{1} - 6x^{3}|^{1} + 7x^{2}|^{1} - x^{1}|^{1}\\
x^3 &= x^{3}|^{1} - 3x^{2}|^{1} + x^{1}|^{1}\\
x^2 &= x^{2}|^{1} - x^{1}|^{1}\\
x &= x^{1}|^{1}
\end{aligned}$$

So that the integral expression becomes

$$\begin{array}{l|l|l|l|l|l}
24\,\dfrac{(x-1)^{6}|^{1}}{6} - 240 & \dfrac{(x-1)^{5}|^{1}}{5} + 600 & \dfrac{(x-1)^{4}|^{1}}{4} - 360 & \dfrac{(x-1)^{3}|^{1}}{3} + 24 & \dfrac{(x-1)^{2}|^{1}}{2} & \\
\qquad + 18 & \qquad -108 & \qquad +126 & \qquad -18 & & \\
 & \qquad -\ 3 & \qquad +\ 9 & \qquad -\ 3 & & +7x \\
 & & \qquad +\ 50 & \qquad -50 & & +C \\
 & & & \qquad -20 & &
\end{array}$$

$$= 24\,\frac{(x-1)^{6}|^{1}}{6} - 222\,\frac{(x-1)^{5}|^{1}}{5} + 489\,\frac{(x-1)^{4}|^{1}}{4}$$
$$- 175\,\frac{(x-1)^{3}|^{1}}{3} - 67\,\frac{(x-1)^{2}|^{1}}{2} + 7x + C.$$

If this had been done by the method given in Pearson's *Finite Differences* (Art. 30), we should have had to solve 6 equations containing 6 unknown quantities.

If we wished to express $(x+a_1)(x+a_2)\ldots(x+a_n)$ under the form $A_n x^{n}|^{-1} + B_n x^{n-1}|^{-1} + \&c.$ our tables would be

I	II	III	IV	V
· a_1 ·	a_2 ·	a_3 ·	a_4 ·	a_5
·	·	·	·	·
	· a_2+1 ·	a_3+1 ·	a_4+1 ·	a_5+1
	·	·	·	·
		· a_3+2 ·	a_4+2 ·	a_5+2
		·	·	·
			· a_4+3 ·	a_5+3
			·	·
				· a_5+4
				·

I	B	II	P	C	III	Q	D	IV	R	E	V	S	F
a_1	B_1	a_2	B_1a_2	C_1	a_3	C_1a_3	D_1	a_4	D_1a_4	E_1	a_5	E_1a_5	F_1
a_2+1	B_2	a_3+1	$B_2(a_3+1)$	C_2	a_4+1	$C_2(a_4+1)$	D_2	a_5+1	$D_2(a_5+1)$	E_2			
a_3+2	B_3	a_4+2	$B_3(a_4+2)$	C_3	a_5+5	$C_5(a_5+2)$	D_3						
a_4+3	B_4	a_5+2	$B_4(a_5+2)$	C_4									
a_5+4	B_5												

If we wished to arrange $(x+a_1)(x+a_2) \ldots (x+a_n)$ according to the powers of x, we should have to use the following table:

	B		P	C		Q	D		R	E		S	F
a_1	B_1	a_2	B_1a_2	C_1	a_3	C_1a_3	D_1	a_4	D_1a_4	E_1	a_5	E_1a_5	F_1
a_2	B_2	a_3	B_2a_3	C_2	a_4	C_2a_4	D_2	a_5	D_2a_5	E_2			
a_3	B_3	a_4	B_3a_4	C_3	a_5	C_3a_5	D_3						
a_4	B_4	a_5	B_4a_5	C_4									
a_5	B_5												

The methods given in this and the following article are particular cases of methods given by Nicholson.

14. To transform the fraction

$$\frac{1}{(x+a_1)(x+a_2)\ldots(x+a_n)} \text{ into the series } \frac{A_0}{x^{n|1}} + \frac{A_1}{x^{n+1|1}} + \&c.$$

Assume

$$\frac{1}{(x+a_1)(x+a_2)\ldots(x+a_{n-1})} = \frac{A_{n-1}}{x^{n-1|1}} + \frac{B_{n-1}}{x^{n|1}} + \frac{C_{n-1}}{x^{n+1|1}} + \&c.$$

Then since

$$\begin{aligned} x+a_n &= x+n-1+a_n-(n-1) \\ &= x+n \quad +a_n-n \\ &= x+n+1+a_n-(n+1) \\ &= \&c., \end{aligned}$$

$$\begin{aligned} \frac{1}{(x+a_1)\ldots(x+a_n)} &= \frac{A_{n-1}}{x^{n-1|1}\{x+n-1+a_n-(n-1)\}} \\ &\quad + \frac{B_{n-1}}{x^{n|1}(x+n+a_n-n)} \\ &\quad + \frac{C_{n-1}}{x^{n+1|1}(x+n+1+a_n-n+1)} + \&c. \\ &= \frac{A_n}{x^{n|1}} + \frac{B_n}{x^{n+1|1}} + \frac{C_n}{x^{n+2|1}} + \&c. \end{aligned}$$

These two series will be rendered identical by the assumptions $A_n = A_{n-1}$, $B_n = -A_n\{a_n-(n-1)\}+B_{n-1}$, $C_n = -B_n(a_n-n)+C_{n-1}$, and so on, for in that case we should have

$$\frac{A_{n-1}}{x^{n-1|1}\{x+n-1+a_n-(n-1)\}} - \frac{A_n}{x^{n|1}} = \frac{-A_n\{a_n-(n-1)\}}{x^{n|1}(x+a_n)},$$

$$\frac{-A_n\{a_n-(n-1)\}+B_{n-1}}{x^{n|1}(x+n+a_n-n)} - \frac{B_n}{x^{n+1|1}} = \frac{B_n}{x^{n+1|1}(x+a_n)},$$

$$\frac{-B_n(a_n-n)+C_{n-1}}{x^{n+1|1}\{x+n+1+a_n-(n+1)\}} - \frac{C_n}{x^{n+2|1}} = \frac{C_n}{x^{n+2|1}(x+a_n)},$$

and so on.

Each of the quantities A will equal 1 as may be seen from the series

$$1 = \frac{A_0}{x^{0|1}} + \frac{B_0}{x^{1|1}} + \&c.,$$

which is satisfied by $A_0 = 1$, $B_0 = C_0 = \&c. = 0$.

Our tables will be

I				
	· $-a_1$	· $-a_2$	$-a_3$	$-a_4$
II	·	·		
	· $1-a_1$	. $1-a_2$	. $1-a_3$	$1-a_4$
III	·	·	·	
	· $2-a_1$	. $2-a_2$	. $2-a_3$	. $2-a_4$
	·	·	·	·
		. $3-a_2$	. $3-a_3$	. $3-a_4$
		·	·	·
			. $4-a_3$	. $4-a_4$
			·	·
				. $5-a_4$
				·

	I	B	II	P	C	III	Q	D
(1)	$0-a_1$	B_1	$1-a_1$	$B_1(1-a_1)$	C_1	$2-a_1$	$(2-a_1)C_1$	D_1
(2)	$1-a_2$	B_2	$2-a_2$	$B_2(2-a_2)$	C_2	$3-a_2$	$(3-a_2)C_2$	D_2
(3)	$2-a_3$	B_3	$3-a_3$	$B_3(3-a_3)$	C_3	$4-a_3$	$(4-a_3)C_3$	D_3
(4)	$3-a_4$	B_4	$4-a_4$	$B_4(4-a_4)$	C_4	$5-a_4$	$(5-a_4)C_4$	D_4

The same law holds as in the other cases.

If we had to transform it into the form $\frac{A_n}{x^{n|-1}} + \frac{B_n}{x^{n+1|-1}} +$ &c. the table would be

	I	B	II	P	C	III	Q	D
(1)	$-a_1$	B_1	$-(a_1+1)$	P_1	C_1	$-(a_1+2)$	Q_1	D_1
(2)	$-(a_2+1)$	B_2	$-(a_2+2)$	P_2	C_2	$-(a_2+3)$	Q_2	D_2
(3)	$-(a_3+2)$	B_3	$-(a_3+3)$	P_3	C_3	$-(a_3+4)$	Q_3	D_3
(4)	$-(a_4+3)$	B_4	$-a_4+4$	P_4	C_4	$-(a_4+5)$	Q_4	D_4
(5)	$-(a_5+4)$	B_5	$-a_5+5$	P_5	C_5	$-(a_5+6)$	Q_5	D_5
(6)	$-a_6+5$	B_6	$-a_6+6$	P_6	C_6	$-(a_6+7)$	Q_6	D_6

Example $$\frac{1}{(x-1)(x-3)(x-5)},$$

-1	-3	-5
0	-2	-4
$+1$	-1	-3
2	0	-2
	-1	-1
		0

I	B	II	P	C	III	Q	D	IV	R
1	1	0	0	0	-1	0	0	0	0
2	3	1	3	3	0	0	0	-1	0
3	6	2	12	15	$+1$	15	15	0	0

Answer $$\frac{1}{x^{3\|-1}} + \frac{6}{x^{4\|-1}} + \frac{15}{x^{5\|-1}} + \frac{15}{x^{6\|-1}}.$$

If we wished our series to proceed according to negative powers we should have to form a table as follows:

	B	P	C	P	D
$-a_1$	B_1	$-a_1B_1$	C_1	$-a_1C_1$	D_1
$-a_2$	B_2	$-a_2B_2$	C_2	$-a_2C_2$	D_2
$-a_3$	B_3	$-a_3B_3$	C_3	$-a_3C_3$	D_3
$-a_4$	B_4	$-a_4B_4$	C_4	$-a_4C_4$	D_4

$$\frac{1}{(x-1)(x+3)(x+5)} = \frac{1}{x^3} - \frac{7}{x^4} + \frac{42}{x^5} - \frac{230}{x^6} + \dots,$$

	B	P	C	Q	D
1	1	1	1	1	1
-3	-2	6	7	-21	-20
-5	-7	35	42	-210	-230

15. Some of the following formulas put the analogy between factorials and indices in a striking light:

$$F(x\Delta)\, x^{m|1} = F(m)\, x^{m|1}, \quad F(xD)\, x^m = F(m)\, x^m \ldots\ (1),$$

$$(x\Delta)^{n|-1} u = x^{n|1} \Delta^n u, \qquad (xD)^{n|-1} u = x^n D^n u \ldots\ (2),$$

$$F(x\Delta) x^{m|1} v = x^{m|1} F(x\Delta + mE)\, v, \quad F(xD) x^m v = x^m F(D+m)\, v \ldots\ (3).$$

The first two of which are given in Mr. Carmichael's *Calculus of Operations.*

If $U = A_0 + A_1 x^{1|1} + A_2 x^{2|1} + \ldots,$

it follows from (1) that

$$F(x\Delta)\, U = F(0)\, A_0 + F(1)\, A_1 x^{1|1} + F(2)\, A_2 x^{2|1} + \text{\&c.} \ldots (4),$$

$$\frac{1}{F(x\Delta)} U = \frac{1}{F(0)} A_0 + \frac{1}{F(1)} A_1 x^{1|1} + \frac{1}{F(2)} A_2 x^{2|1} + \text{\&c.}$$

Analogous to the formulas

$$F(xD)\, V = F(0)\, B_0 + F(1)\, B_1 x + F(2)\, B_2 x^2 + \text{\&c.},$$

$$\frac{1}{F(xD)} V = \frac{1}{F(0)} B_0 + \frac{1}{F(1)} B_1 x + \frac{1}{F(2)} B_2 x^2 + \text{\&c.},$$

where $\qquad V = B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \text{\&c.}$

16. Any equation of differences of the form

$$A x^{\alpha|1} \Delta^\alpha u + B x^{\beta|1} \Delta^\beta u + \text{\&c.} = P x^{p|1} + Q x^{q|1}$$

can be immediately solved by the formulas of the last article, for it becomes

$$A\, (x\Delta)^{\alpha|1} u + B\, (x\Delta)^{\beta|1} u + \text{\&c.} = P x^{p|1} + Q x^{q|1},$$

i.e. $\quad F(x\Delta)\, u = P x^{p|1} + Q x^{q|1},$

$$u = \frac{1}{F(x\Delta)} (P x^{p|1} + Q x^{q|1}) + \frac{1}{F(x\Delta)}\, 0$$

$$= \frac{P}{F(p)} x^{p|1} + \frac{Q}{F(q)} x^{q|1} + C_a x^{a|1} + C_b x^{b|1} + \text{\&c.},$$

where a, b, &c. are the real unequal roots of the equation

$$F(z)=0.$$

Mr. Carmichael gives the solution of

$$Ax^{x}|^{1}\Delta^{a}u+\ldots=0.$$

Let the proposed example be

$$x^{3}|^{1}\Delta^{3}u-4x^{2}|^{1}\Delta^{2}u-2x^{1}|^{1}\Delta u+4=x^{4}+7x^{3}+14x^{2}+8x,$$

or $(x\Delta)^{3}|^{1}u-4(x\Delta)^{2}|^{1}u-2x\Delta u+4=x^{4}+7x^{3}+14x^{2}+8x,$

Now transforming factorials into powers on the left-hand side of the equation and powers into factorials on the right, by the methods given in Art. (13), we have

$$(x\Delta)^{3}u-(x\Delta)^{2}u-4(x\Delta)u+4=x^{4}|^{1}+x^{3}|^{1}.$$

Now the roots of $z^{3}-z^{2}-4z+4=0$ are $+1$, $+2$, -2, so that we have

$$u=\frac{1}{F(x\Delta)}\{x^{4}|^{1}+x^{3}|^{1}\}+\frac{1}{F(x\Delta)}0$$

$$=\frac{1}{F(4)}x^{4}|^{1}+\frac{1}{F(3)}x^{3}|^{1}+Ax^{1}|^{1}+Bx^{2}|^{1}+Cx^{-2}|^{1}$$

$$=\frac{x^{4}|^{1}}{36}+\frac{x^{3}|^{1}}{10}+Ax+Bx^{2}|^{1}+C\frac{1}{(x-1)^{2}|^{-1}}.$$

17. Any expression, each of whose terms is of the form $Cx^{a}|^{1}y^{\beta}|^{1}z^{\gamma}|^{1}\ldots$, where $a+\beta+\gamma+\ldots=m$ is called a function of the m^{th} factorial degree, or more shortly a homogeneous factorial function.

Let u_m be such a function, then

$$(x\Delta_x+y\Delta_y+z\Delta_z+\ldots)u_m=\Sigma(x\Delta_x+y\Delta_y+z\Delta_z+\ldots)Cx^{a}|^{1}y^{\beta}|^{1}z^{\gamma}|^{1}$$

$$=(a+\beta+\gamma)\,\Sigma Cx^{a}|^{1}y^{\beta}|^{1}z^{\gamma}|^{1}$$

$$=mu_m,$$

or writing ∇ for $x\Delta_x+y\Delta_y+z\Delta_z+\ldots$,

$$\nabla u_m=mu_m,$$

and generally $$F(\nabla)u_m=F(m)u_m.$$

Now if U be any algebraical function it can always be put under the form

$$U = u_0 + u_1 + \ldots + u_m;$$

therefore $F(\nabla)U = F(0)\,u_0 + F(1)\,u_1 + \ldots + F(m)\,u_m,$

$$F\left(\frac{1}{\nabla}\right)U = \frac{1}{F(0)}u_0 + \frac{1}{F(1)}u_1 + \ldots + \frac{1}{F(m)}u_m.$$

If in these three formulas ∇ stood for $xD_x + yD_y + zD_z + \ldots$ and u_m meant a function of m dimensions, we should have known formulas of the Differential Calculus.

18. Since

$$(1+a)^{\nabla} = (1+a)^{x\Delta_x}(1+a)^{y\Delta_y}(1+a)^{z\Delta_z}.$$

It follows (expanding and equating coefficients of like powers of a) that

$$\frac{\nabla^n|^{-1}}{1^n|^1}U = \Sigma\left\{\frac{(x\Delta_x)^{\alpha}|^{-1}}{1^{\alpha}|^1}\frac{(y\Delta_y)^{\beta}|^{-1}}{1^{\beta}|^1}\frac{(z\Delta_z)^{\gamma}|^{-1}\ldots}{1^{\gamma}|^1}\right\}U,$$

where $\qquad \alpha + \beta + \gamma + = n.$

But since $\qquad (x\Delta_x)^{\alpha}|^{-1} = x^{\alpha}|^1\Delta_x^{\alpha}.$

It follows that

$$\frac{\nabla^n|^{-1}}{1^n|^1}U = \Sigma\,\frac{(x^{\alpha}|^1\Delta_x^{\alpha}.y^{\beta}|^1\Delta_y^{\beta}.z^{\gamma}|^1\Delta_z^{\gamma}\ldots)}{1^{\alpha}|^1\quad 1^{\beta}|^1\quad 1^{\gamma}|^1\ldots}U,$$

or if ∇_n denote the operating symbol formed by changing any term $Ca^{\alpha}b^{\beta}c^{\gamma}\ldots$ in the expansion of $(a+b+c+\ldots)^n$ into $Cx^{\alpha}|^1\Delta_x^{\alpha}\,y^{\beta}|^1\Delta_y^{\beta}\ldots.$ It follows that

$$\nabla^n|^{-1}U = \nabla_n U.$$

All equations of partial differences of the form

$$A\nabla_{\alpha}u + B\nabla_{\beta}u + C\nabla_{\gamma}u\ldots = \Omega$$

may obviously be put under the form

$$\{A\nabla^{\alpha}|^{-1} + B\nabla^{\beta}|^{-1} + C\nabla^{\gamma}|^{-1} + \ldots\}\,u = \Omega.$$

The solution will be given by the evaluation of the symbolic form

$$u = \frac{1}{F(\nabla)}\Omega + \frac{1}{F(\nabla)}0.$$

Now the value of the first term of this solution is perfectly definite, and can be had at once by the formula of Art. 18. If the roots of the equation

$$z^{\alpha}|^{-1} + \frac{B}{A} z^{\beta}|^{-1} + \frac{C}{A} z^{\gamma}|^{-1} + \ldots = 0$$

are all real and unequal, the arbitrary form of the solution is of the form

$$u_a + u_b + u_c + \ldots + u_i,$$

where a, b, $c \ldots i$, are the values of the roots, and u_a, $u_b \ldots i$ are homogeneous factorial functions of the degrees a, b, $c \ldots i$, whose forms are arbitrary. We see here that the number and character of arbitrary functions in a solution are unaffected by the number of independent variables that the equation may contain, and are dependent solely on its order.

The reader is requested to compare Mr. Carmichael's *Calculus of Operations*, p. 35. He will see that the very same symbols interpreted differently give the solution of a partial differential equation.

Example.

$$x^2|^1\Delta_x^{\,2} z + 2xy\Delta_x\Delta_y z + y^2|^1\Delta_y^{\,2} z = \Theta_m + \Theta_n,$$

where Θ_m, Θ_n are given homogeneous factorial functions of the m^{th} and n^{th} degrees respectively.

This becomes

$$\nabla_2 z = \Theta_m + \Theta_n.$$

Hence

$$z = \frac{1}{\nabla^2|^{-1}} (\Theta_m + \Theta_n) + \frac{1}{\nabla^2|^{-1}}\, 0$$

$$= \frac{\Theta_m}{m^2|^{-1}} + \frac{\Theta_n}{n^2|^{-1}} + u_0 + u_1.$$

NOTE ON THE CONDITIONS THAT THE EQUATION OF THE SECOND DEGREE SHOULD REPRESENT A CIRCLE OR SPHERE.

By N. M. Ferrers.

I propose in this note to investigate the conditions that the general equation of the second degree between α, β, γ, may represent a circle, and that that between α, β, γ, δ, may represent a sphere. Commencing with the former case, and considering the coordinates α, β, γ of a point P to denote the ratios of the areas of the triangles PBC, PCA, PAB respectively, to that of the triangle of reference ABC, suppose the curve represented by the equation,

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2A'\beta\gamma + 2B'\gamma\alpha + 2C'\alpha\beta = 0,$$

to cut the side BC in the points a_1, a_2, CA in b_1, b_2, AB in c_1, c_2. Then

$$Ab_1 . Ab_2 = Ac_1 . Ac_2.$$

Now $\frac{Ab_1}{b} = \frac{Ab_1}{CA} = \gamma_1$, if γ_1 be the value of γ corresponding to c_1,

and $\frac{Ab_2}{b} = \gamma_2$ .... γ_2..c_2.

Now at every point in the line CA, we have

$$A\alpha^2 + C\gamma^2 + 2B'\gamma\alpha = 0,$$

$$\gamma + \alpha = 1,$$

whence $\quad A(1-\gamma)^2 + C\gamma^2 + 2B'\gamma(1-\gamma) = 0,$

or $\quad (A + C - 2B')\gamma^2 + 2(B' - A)\gamma + A = 0;$

therefore $$\gamma_1\gamma_2 = \frac{A}{A + C - 2B'},$$

and $$Ab_1 . Ab_2 = \frac{A}{A + C - 2B'}\, b^2,$$

similarly $$Ac_1 . Ac_2 = \frac{A}{A + B - 2C'}\, c^2,$$

whence, since $Ab_1 . Ab_2 = Ac_1 . Ac_2$,

we find
$$\frac{C+A-2B'}{b^2} = \frac{A+B-2C'}{c^2}$$
$$= \frac{B+C-2A'}{a^2} \text{ by symmetry,}$$

as necessary conditions that the given equation should represent a circle; and, since they are two in number, they are sufficient.

Hence the conditions that the equation,

$$A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + 2[ab]\alpha\beta + 2[cd]\gamma\delta + 2[ac]\alpha\gamma + 2[bd]\beta\delta$$
$$+ 2[ad]\alpha\delta + 2[bc]\beta\gamma = 0,$$

may represent a sphere, are readily deduced. For if $ABCD$ be the tetrahedron of reference, then supposing P to be any point in the face BCD, we shall have

$$\Delta PCD : \Delta PDB : \Delta PBC :: \beta : \gamma : \delta.$$

And the section of a sphere by a plane is a circle, hence, (CD) denoting the length of the edge CD, &c.,

$$\frac{C+D-2[cd]}{(CD)^2} = \frac{D+B-2[bd]}{(BD)^2} = \frac{B+C-2[bc]}{(BC)^2}$$
$$= \frac{A+B-2[ab]}{(AB)^2} = \frac{A+C-2[ac]}{(AC)^2} = \frac{A+D-2[ad]}{(AD)^2},$$

which conditions, being five in number, are sufficient.

It hence follows that the equation to the sphere circumscribed about $ABCD$ is

$$(AB)^2\alpha\beta + (CD)^2\gamma\delta + (AC)^2\alpha\gamma + (BD)^2\beta\delta + (AD)^2\alpha\delta + (BC)^2\beta\gamma = 0.$$

ON THE MULTIPLE INTEGRAL $\int^n dxdy\ldots dz$, WHOSE LIMITS ARE $p_1=a_1x+b_1y+\ldots+h_1z>0$, $p_2>0$, $\ldots p_n>0$, AND $x^2+y^2+\ldots+z^2<1$.

By Dr. SCHLAEFLI, Professor of Mathematics at the University of Bern.

SECTION I.

LIMITED merely by the last inequality, this integral (exhibiting for $n=2$, 3 the area of a circle or the volume of a sphere with the radius 1) was, long ago, ascertained to be $\pi^{\frac{n}{2}} : \Gamma\left(\frac{n}{2}+1\right)$.* But I do not know that the similar generalization of the integral representing a sector of a circle or a spherical sector on a triangular base has hitherto treated of. For this purpose, there must besides be given n limits of the linear and homogeneous form $p>0$. For if the number of such limits were less than n, the integral can quite easily be reduced to a number of integrations equal to that of the linear limits given; and if there were more than n such limits, the integral may be resolved into several others having each only n such limits. We shall therefore here confine our attention principally to the integral with n linear limits.

A short view of its properties may previously be given.

1°. The number of integrations to be performed can be reduced to $\frac{n}{2}$ or $\frac{n-1}{2}$, according as n is even or odd.

2°. The number of independent elements, as a function of which the value of the finite integral is to be looked on, is $\frac{n(n-1)}{2}$; for if

$$p=ax+by+\ldots>0, \quad p'=a'x+b'y+\ldots>0,$$

where $a^2+b^2+\ldots=1$, $a'^2+b'^2+\ldots=1$, be any two of the given limits, we may define $aa'+bb'+\ldots$ to be one of these elements. But, moreover, the proposed integral can, in n different ways, be divided into $1.2.3\ldots(n-1)$ similar functions of such a special character that each has only $(n-1)$ of its elements connected by algebraical relations with those of the proposed function, while all the rest are equal to zero.

* See a paper by M. Catalan in Liouville's *Journal*, t. VI. p. 81, where he refers to t. IV. p. 336.

3°. For every value of n greater than 2, there are a determinate number of cases where this function at the same time becomes a *rational* part of the total integral $\pi^{\frac{n}{2}} : \Gamma\left(\frac{n}{2}+1\right)$, while each of its $(n-1)$ elements is the cosine of a *rational* part of π. In this view the number $n=4$ yields the greatest variety of possible cases; and to give the reader a previous idea of the strangeness of the subject, I may mention that the total integral $\int^n dwdxdydz \; (w^2+x^2+y^2+z^2<1) = \frac{1}{2}\pi^2$ can be divided into 14400 *superposable* parts by means of linear and homogeneous limits. The word "superposable" involves the conception of linear and homogeneous substitutions such as do not change the form of the sum of squares of the four variables, while the effect of these substitutions consists in bringing any two of those parts to a complete formal coincidence. It will be readily perceived that for $n=3$ the regular polyhedrons fall under this point of view.

SECTION II.

We require some previous definitions.

One equation or any number of equations, not above that of the variables, being given, we are wont to call *solution* any set of values of the variables which satisfies the system of equations; and if we conceive all such solutions together, provided the system does not determine the values of the variables, we shall have the idea of a *continuum*, in which every solution is closely surrounded by other solutions, so that we may continuously pass from one of any two different solutions to the other through mere solutions. Such a set of solutions forms a *simple continuum* which we may call *a curve*, whereas the whole continuum is to be called an m-fold one when the number of variables exceeds by m that of the equations. But we may also be allowed, abstracting from any given condition to embrace in one conception all the sets of values of the variables, and then, against the ordinary practice, to continue using the term *solution*. And the whole of such solutions not subject to any condition, if there be n variables, may then be called the n-fold *totality*.* Thus a single solution and the totality are the utmost ends of a row, at the intermediate stages of which range the different sorts of continua,

* The n-fold totality is in fact space of n dimensions, a solution is a point in such space, and the intermediate continua of 1, 2,... or $(n-1)$ equations are the loci which in such space correspond to surfaces and curves in ordinary space of three dimensions.—A. C.

according as they are represented by 1 or 2 or 3, ... or $n-1$ equations.

We will also use the term *coordinates* instead of the cumbrous one "values of the variables belonging to some solution."

As two solutions may more or less deviate from each other, and as the set of differences between their like-named coordinates, changing as it does with every linear transformation, does not give us a proper and consistent idea of the deviation being in itself a single thing, we may say that the square root of a quadratic and homogeneous function of those differences, such as never to vanish for real values of them, shall represent the deviation or *distance* between the two solutions. Again, since, by the help of linear and real substitutions, such a quadratic function can always be transformed into a sum of the squares of the differences of coordinates, we will distinguish any coordinate system in which this most simple expression of *distance* takes place from any other in which it is not so, by calling the former a *system of orthogonal coordinates.* Accordingly, every system of linear substitutions for the differences of orthogonal coordinates, which does not interfere with the above condition of orthogonality, may be called an *orthogonal transformation.* The matrice of the coefficients of substitution, has then the well known properties; 1°. that its determinant is absolutely equal to unity, we add, to the *positive* unity, since we may always arrange the new variables so as to effect this result; 2°. that its minors are equal to the corresponding primary elements, in such manner that the matrice turned about its positive diagonal becomes that of the inverse substitution. Other properties are evident from the definition itself. The subject of this paper requires that we shall mostly employ orthogonal coordinates.

Again, we shall call a *linear continuum* every one that is merely determined by linear equations, whose number, of course, is less that n, the order of the totality. Let this number be $n-m$, we shall then say such a linear continuum sometimes to be one of $n-m$ *conditions*, sometimes to be one of m *dimensions* or an m-fold one, the latter predicate pointing out the number of independent variables. As the two cases of one condition and of one dimension will often occur, we want shorter denominations for both. In the former case we shall use indifferently the words *equation*, *polynome*, *continuum*, when no mistake is to be feared; in the latter case we shall adopt the word *line* or *radius*, this latter in particular when the lines considered depart from a common solution. In general, we shall not shun employing the language of geometry, whenever it contributes to shortness and clearness. As for instance, if a system of equations (whether linear or not) be satisfied by every solution of another system of a greater number of equations, we shall then say that the former continuum *passes through* the latter, or that the latter *lies in* (or *upon*) the former. This same idea may also at times be expressed by the notion of *dependency;* so an equation (whether linear or not) is said to *depend* upon several others, when it is satisfied by every solution of the system of all the latter, that is, when the former continuum of one condition passes through the latter continuum of more conditions than one. Moreover, at least as regards linear continua, we shall use this word *dependency* in a still larger sense. A linear m-fold continuum is determined by $m+1$ *independent* solutions; now if an $(m+2)^{\text{th}}$ solution lie upon that continuum, we shall say this new solution to *depend* upon those $m+1$ solutions; and hereby the meaning of the word *independent* just employed has also been explained. The primary importance of this notion obliges us some longer to dwell upon it. Let us conceive $n-m$ equations linear and homogeneous with respect to the $n+1$ variables $w, x, y, \dots z$; if we then, at haphazard, take $m+1$ solutions $(a_0, b_0, c_0, \dots h_0)$, $(a_1, \dots)$, $\dots (a_m, \dots)$ of this system, it will be easily seen, that there is no other solution of the same system but can be brought into the form $w = a_0 t_0 + a_1 t_1 + \dots + a_m t_m$, $x = b_0 t_0 + \dots$, $\dots z = h_0 t_0 + \dots$, where $t_0, t_1, \dots t_m$ denote arbitrary factors, the m ratios of which are very proper to put in evidence the m dimensions of the continuum under consideration; for we ought to have subjoined that the n ratios $\frac{x}{w}, \frac{y}{w}, \dots \frac{z}{w}$ represented coordinates. In an other view, if a linear continuum of one condition $q = 0$ passes through, or depends upon, one of m conditions $p_1 = 0$, $p_2 = 0$, $\dots p_m = 0$, it then necessarily follows that $q = a_1 p_1 + \dots + a_m p_m$, where $a_1, a_2, \dots a_m$ denote any constants. In one word, dependency and linear expressibility are exchangeable notions both for solutions and for polynomes. On this occasion it is proper to remark that in the n-fold totality there cannot be more independent (linear) polynomes than $n+1$.

When all that falls under consideration is passing in an m-fold linear continuum, then by a suitable transformation of coordinates we can make m of them disappear from the $n-m$ equations; and since the $n-m$ remaining coordinates must therefore vanish, we shall keep only m free variables, and so, in fact, all will be just as if we were merely concerned with an m-fold totality. On the other hand, if it be not the continuum itself, but the mutual relation or *position* of the linear equations determining it which is to be considered, we may then neglect all the m disappearing variables and deal only with the $n-m$ variables, the vanishing of which determines the continuum. Both these points of view will be often employed hereafter, and it imports much to distinguish them well.

Let r be the distance of two solutions O, P, and $a, b, \ldots h$ the orthogonal coordinates of P, if O be taken for the *origin;* having then $r^2 = a^2 + b^2 + \ldots + h^2$, we may define $a, b, \ldots h$ to be the *projections* of the radius r. This limited line OP, having r for its length, contains all solutions ($x = at$, $y = bt$, ... $z = ht$) for which $0 < t < 1$. Now let P' be a third solution, independent of O and P, give it the coordinates a', b', ... h', and call r' its distance from O. As then the three solutions determine a linear double continuum (a plane), we may adopt the geometrical term *angle* in its proper meaning, in order to indicate the mutual position of the two unlimited lines OP, OP', and on having duly transformed the coordinates and again restored the primitive ones, we shall find

$$rr' \cos\theta = aa' + bb' + \ldots + hh',$$

where θ denotes the angle included by the two lines.

A linear continuum of one condition

$$p = ax + by + \ldots + hz + k = 0$$

has two sides corresponding to the inequalities $p > 0$ and $p < 0$. Though it be indifferent, in general, by what constant factor soever we multiply the equation, yet, as regards the two sides, we must distinguish between a positive and negative factors. To escape all ambiguity we will always suppose the coefficients such as to make $a^2 + b^2 + \ldots + h^2 = 1$ and reject the idea of a multiplication by -1, unless we should want to interchange the two sides. It is then easy to show that, if $(x, y, \ldots z)$ be any solution P of the totality and $(x', y', \ldots z')$ a solution P' of the given equation, the radius $r = \sqrt{\{(x'-x)^2 + (y'-y)^2 + \ldots + (z'-z)^2\}}$ becomes a minimum when $x' = x - ap$, $y' = y - bp$, ... $z' = z - hp$. In this case r is $= p$ or $= -p$, according as p is positive or negative, that is, according as the free solution P lies on the positive or negative side of the equation. The line $(x + ar, y + br, \ldots z + hr)$, where r means an indeterminate length, may then be called a *normal* to the equation, $a, b, \ldots h$ being the *cosines of direction* of this normal. Now if there is given another equation $p' = a'x + b'y + \ldots + h'z + k' = 0$, where also $a'^2 + b'^2 + \ldots + h'^2 = 1$, on having drawn a normal to it, we call the supplement θ of the angle between the two normals the *angle included* by the two equations $p = 0$, $p' = 0$, and we therefore put $-\cos\theta = aa' + bb' + \ldots + hh'$. The reason why we prefix the negative sign to $\cos\theta$ is that the angle is a kind of simple integral taken between the limits $p > 0$, $p' > 0$, which vanishes only when the limits are $p > 0$, $(-p) > 0$, that is, when the limits are such as not to allow the solution the smallest deviation from the equation $p = 0$. But then we have $a' = -a$, $b' = -b$, ... $h' = -h$, and consequently $\cos\theta = a^2 + b^2 + \ldots + h^2 = 1$, $\theta = 0$, which agrees with the above statement.

An m-fold and $(n-m)$-fold linear continua may be called *normal* to each other, when there is a system of orthogonal coordinates such that the former continuum comes to be represented by the equations $x_{m+1} = 0$, $x_{m+2} = 0$, ... $x_n = 0$, and the latter by the equations $x_1 = 0$, $x_2 = 0$, ... $x_m = 0$, where x_1, x_2, ... x_n denote the coordinates. Every two lines, one lying in the former and the other in the latter continuum, will then make a right angle; and so will every two linear continua of one condition passing respectively through those.

A few words may suffice to show how we are to judge of the mutual position of two linear continua, being respectively of p and q dimensions. If $p + q > n$, then $(n-p) + (n-q) < n$, that is, the $2n - p - q$ equations of both continua determine a linear continuum of $p + q - n$ dimensions as their common intersection. We then orthogonally transform the coordinate system so as to have $p + q - n$ of its axes in this intersection, and, neglecting the corresponding dimensions, we are only concerned with a totality of $2n - p - q$ dimensions, and the given continua will be replaced respectively by one of $n - q$ and another of $n - p$ dimensions. Thus the question is reduced to the judgment of the mutual position of two continua whose numbers of dimensions are together equal to that of the totality. On the other hand, if $p + q < n$, we make to pass through a common origin two continua *parallel* to the given ones; they will determine together a linear continuum of $p + q$ dimensions, which we regard as a totality; and the question is reduced to the same case as before. Now if $p + q = n$ and $p > q$, we conceive an orthogonal coordinate system having q of its axes in the second continuum; then the remaining p axes will determine a linear continuum which intersects the first continuum in one such of $n - 2q = p - q$ dimensions; in this we assume $p - q$ of the remaining p axes.

Causing then the corresponding dimensions to disappear, we are concerned with a totality of only $2q$ dimensions and have reduced the first continuum to q dimensions, while the second has kept its complete number q of dimensions. And so at all events we are lastly led to consider two linear continua each of q dimensions, having one common solution only and situated in a totality of $2q$ dimensions. I now assert that, in general, it is always possible to draw in each continuum from the common origin q axes orthogonal among themselves, and such that each of them makes an oblique angle with the corresponding axis in the other continuum, but right angles with the $q-1$ other axes of this other continuum. Thus the mutual position of the two linear continua merely depends upon the values of the q oblique angles.

In order to complete what has been said, we must remark that any q independent lines, departing from a common solution and situated in a q-fold linear continuum, may be regarded as its *axes*, for that every other line in it, which also departs from that common solution, can be simply represented by linear compositions of these axes.

Conceiving all coordinates to be fractions with a common denominator w (to be regarded in a manner as the $(n+1)^{\text{th}}$ coordinate), we may call *parallel* any two linear equations which cannot consist with each other unless w be $=0$, that is, which have their complete intersection lying in the *infinitely distant equation* $w=0$, or in a word, which differ only by the constant term. Two lines are *parallel* when determined by $(n-1)$ pairs of parallel equations, or, what is the same thing, when the homologous projections of limited portions of them are proportional.

The general idea which in the n-fold totality answers to the area in plane and to the volume in space may be termed *measure*, and for a wholly limited portion of the totality we assume as the unit the entirety of the solutions comprised in

$$0<x<1,\ 0<y<1,\ \ldots\ 0<z<1.$$

Thus we obtain the integral $\int^n dxdy \ldots dz$ as an expression of every limited portion of the totality.

A limited portion of a linear $(n-m)$-fold continuum must have the same measure as in the $(n-m)$-fold totality, wherefore the most natural way of evaluating it will be that we orthogonally transform the coordinates so that the m equations of the continuum contain no more than m coordinates; then if $x_{m+1}, x_{m+2}, \ldots x_n$ be the other coordinates (disappearing from the equations), the integral $\int^{n-m} dx_{m+1} dx_{m+2} \ldots dx_n$ will be the required measure. Now restoring the primitive orthogonal system, let

$$\overset{i}{p} = \overset{i}{a}_1 x_1 + \overset{i}{a}_2 x_2 + \ldots + \overset{i}{a}_n x_n = 0, \quad [i = 1, 2, \ldots m]$$

be the equations of the continuum and M the measure of its portion, form all the determinants of the rectangular matrice $\left\| \begin{matrix} \overset{i}{a}_1, \overset{i}{a}_2, \ldots \overset{i}{a}_n \\ [i = 1, 2, \ldots m] \end{matrix} \right\|$ and let R^2 denote the sum of their squares, we shall then have as many equations of the form

$$R \int^{n-m} dx_1 dx_2 \ldots dx_{n-m} = \begin{vmatrix} \overset{1}{a}_{n-m+1}, & \overset{1}{a}_{n-m+2}, & \ldots \overset{1}{a}_n \\ \overset{2}{a}_{n-m+1}, & \overset{2}{a}_{n-m+2}, & \ldots \overset{2}{a}_n \\ \cdots & \cdots & \cdots \\ \overset{m}{a}_{n-m+1}, & \overset{m}{a}_{n-m+2}, & \ldots \overset{m}{a}_n \end{vmatrix} M,$$

as there are determinants in the matrice. Now, since the conception of an infinitely small portion of measure of a *curved* continuum of m conditions does not imply any more difficulty, we may at once state this consequence:

Let $f_1(x_1, x_2, \ldots x_n) = 0$, $f_2 = 0$, $\ldots f_m = 0$ be the m equations of a curved continuum, form all the determinants of the rectangular matrice

$$\left\| \begin{matrix} \frac{df_1}{dx_1}, & \frac{df_1}{dx_2}, & \frac{df_1}{dx_3}, & \ldots \frac{df_1}{dx_n} \\ \frac{df_2}{dx_1}, & \frac{df_2}{dx_2}, & \frac{df_2}{dx_3}, & \ldots \frac{df_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{df_m}{dx_1}, & \frac{df_m}{dx_2}, & \frac{df_m}{dx_3}, & \ldots \frac{df_m}{dx_n} \end{matrix} \right\|,$$

call R^2 the sum of their squares and let $V(123\ldots m)$, for instance, denote that determinant which answers to the chosen combination of indices of coordinates; then the measure of a limited portion of the continuum will be

$$\int^{n-m} \frac{R}{V(123\ldots m)} dx_{m+1} dx_{m+2} \ldots dx_n.$$

For a continuum of one condition $f(x_1, x_2, \ldots x_n)=0$, we therefore get among others the expression

$$\int^{n-1}\sqrt{\left\{\left(\frac{df}{dx_1}\right)^2+\left(\frac{df}{dx_2}\right)^2+\ldots+\left(\frac{df}{dx_n}\right)^2\right\}}\,\frac{dx_2\,dx_3\ldots dx_n}{\left(\frac{df}{dx_1}\right)}.$$

An m-fold curved continuum may also be represented by equating the n coordinates to so many functions of the m independent variables $t_1, t_2, \ldots t_m$. Then if R be the square root of the sum of squares of all determinants comprised in the rectangular matrice

$$\left\| \begin{matrix} \frac{dx_1}{dt_i}, \frac{dx_2}{dt_i}, \frac{dx_3}{dt_i}, \ldots \frac{dx_n}{dt_i} \\ [i=1, 2, 3, \ldots m] \end{matrix} \right\|,$$

the measure of the continuum will be $\int^m R\,dt_1\,dt_2\ldots dt_m$. As a particular case we may point out that the measure of a simple continuum (*curve*) is $\int\sqrt{(dx_1^2+dx_2^2+\ldots+dx_n^2)}$.

If $p_1, p_2, \ldots p_{n+1}$ be $(n+1)$ independent linear functions of the n coordinates, there will be an identical equation

$$\alpha_1 p_1+\alpha_2 p_2+\ldots+\alpha_{n+1}p_{n+1}=C,$$

where none of the constant coefficients α vanish; for this is the meaning of the term "independent." The polynomes p being intended to serve as limits, we may for greater simplicity replace αp by p so that the identical equation becomes $p_1+p_2+\ldots+p_{n+1}=C$, where we may suppose the constant C to be positive. Now let us consider the integral $\int^n dx_1\,dx_2\ldots dx_n$ with the limits $p_1>0$, $p_2>0$, $\ldots p_{n+1}>0$; it is evident that no p can surpass C; the proposed integral therefore contains no other elements than such as give all the p's finite values. Again, it must be possible to express the coordinates $x_1, x_2, \ldots x_n$ in terms of all the p's except any one; for if not, the n employed p's would be connected by an identical relation, against the supposed independency. Consequently every solution within the given limits has all its coordinates finite, whence the integral is also finite. It is also obvious that any alteration whatever of the signs of the boundaries would destroy the possibility of enclosing a finite portion of the totality. The final conclusion may be thus stated:

"At least $n+1$ linear equations are required to enclose a finite portion of the totality; for this purpose it is sufficient and necessary that there should be no identical relation containing less than all the polynomes; and the problem is then solved in one way only."

When more than $(n+1)$ linear equations bound a portion of the totality, we still want a detailed statement, by what other equations and how far each single equation is bounded, in a word, we want a knowledge of the *figure* of each bounding equation, and this itself anew requires the same thing for an inferior totality, and so on, till we have attained to the bounding lines or *edges* of the first given portion of the highest totality. We propose to call the bounded portion of the totality *polyschemon*, and the bounding figures of descending orders its *perischemons*, if need be *primary*, *secondary*, ...*n*-ary perischemons, or inversely: perischemons of no dimension (the vertices), of one (the edges), of two (the plane faces), of three (the polyhedrons), of four, ... of $n-1$ dimensions. There arises an inconvenience, it is true, from this mode of expression, namely that the vertices or perischemons of no dimension, in a manner, correspond to the bounding linear equations or primary perischemons, the edges to the secondary ones, and so on. It would perhaps be more suitable to regard the vertices as perischemons of the first ascending order, and so on.

Euler's theorem on polyhedrons still holds good for these polyschemons of the n-fold totality. For if we conceive a coherent net of primary perischemons and let $a_0, a_1, a_2, \ldots a_{n-1}, a_n$ respectively denote the number of the vertices, the edges, the plane faces, the polyhedrons, ... the bounding linear equations, and lastly the number zero or unity, according as the net is open or encloses a true polyschemon, the relation

$$a_0 - a_1 + a_2 - a_3 + a_4 - \ldots + (-1)^{n-1}a_{n-1} + (-)^n a_n = 1$$

will exist. It is to be proved in a successively ascending way.

If we let pass through the origin n linear equations $p_1 = 0,\ p_2 = 0,\ \ldots p_n = 0$ such as not to fulfil any identical relation $\Sigma ap = 0$, the $2n$ limits

$$0 < p_1 < h_1,\ \ 0 < p_2 < h_2,\ \ \ldots 0 < p_n < h_n$$

will enclose a finite portion of totality which we may call a *paralleloschemon*. Its measure is $h_1 h_2 \ldots h_n : \Delta$, if Δ denote the determinant formed by the coefficients in the equations $p = 0$. Or, if $(\overset{1}{a}_1, \overset{1}{a}_2, \ldots \overset{1}{a}_n)$ be the solution of the linear system $p_1 = h_1, p_2 = 0, p_3 = 0, \ldots p_n = 0$, and so on, the measure is also $\begin{vmatrix} \overset{i}{a}_1, \overset{i}{a}_2, \ldots \overset{i}{a}_n \\ [i = 1, 2, \ldots n] \end{vmatrix}$. That is, the measure of a paralleloschemon is equal to the determinant formed by the projections of its n edges departing from one vertex. This proposition can be proved, exactly as in Geometry, by taking away and adjoining superposable portions, always in ascending from a lower totality to the next higher. Calling *base* the measure B_1 of the perischemon $p_1 = 0$ and *height* its distance h_1 from the parallel perischemon, then $B_1 h_1$ is the measure of the paralleloschemon, B_1 being equal to the square root of the sum of squares of the minors corresponding to the first horizontal line of the above matrice. If θ_{12} be the angle between the perischemons $p_1 = 0$ and $p_2 = 0$, and if their common secondary perischemon have the measure C_{12}, the measure of the paralleloschemon will also be $B_1 B_2 \sin\theta_{12} : C_{12}$.

Again, if we let pass through the extremities of the n edges departing from the origin a new equation $p_{n+1} = 0$, we shall have enclosed a polyschemon of the simplest kind, and its measure will be the $(1.2.3\ldots n)$-th part of that of the paralleloschemon. Hence it follows that in general,

$$\overset{i}{a}_1,\ \overset{i}{a}_2,\ \ldots \overset{i}{a}_n\ [i = 1, 2, 3, \ldots n+1]$$

denoting the coordinates of the $n+1$ vertices of such a simplest polyschemon, its measure will be

$$\frac{1}{1.2.3\ldots n} \left| \begin{matrix} 1,\ \overset{i}{a}_1,\ \overset{i}{a}_2,\ \overset{i}{a}_3,\ \ldots \overset{i}{a}_n \\ [i = 1, 2, 3, \ldots n+1] \end{matrix} \right| .$$

Section III.

The first traces that I know of a theory of the general equation of the second degree appear in Laplace's *Mécanique Céleste* on the occasion of the secular perturbations of the solar system. The problem there solved, translated into the language here adopted, is that of finding the principal axes of a quadratic equation in direction and magnitude, when it is written down in orthogonal coordinates and with real coefficients. The answer shows that the directions and squares of the lengths of the principal axes are all real. A perspective treatment of the same subject would show that there are only $\frac{n+2}{2}$ or $\frac{n+1}{2}$ (according as n is even or odd) different kinds of equations, one of which admits of no real solution; the number of the real kinds therefore is $\frac{n}{2}$ or $\frac{n-1}{2}$.

As for example, when $n=3$, there are only two general kinds, the rectilinear one and that which contains no real right line.

We do not dwell further on this general subject, but we proceed to one of its most particular cases, the general spheric equation $x_1^2+x_2^2+\ldots+x_n^2=a^2$; the origin here assumed may be called *centre* and the constant a *radius*. For the curved continuum itself, represented by this equation, I propose the term *polysphere* or n-sphere; thus, a pair of points in a straight line will be a *monosphere*, the circle a *disphere*, and the spherical surface properly so called a *trisphere*. These terms are very harsh, but I do not know how to avoid this.

Although in the title of this paper I have suggested the conception of a portion of the totality, enclosed by the spheric equation and n linear equations passing through its centre, yet by preference it will be the corresponding portion of the polyspheric continuum which comes here under consideration. This, however, is for the present purpose but a slight modification.

Of primary importance is the following transformation of coordinates:

$$x_1\ =r\cos\phi_1,\quad x_2=r\sin\phi_1\cos\phi_2,\quad x_3=r\sin\phi_1\sin\phi_2\cos\phi_3,\ \ldots$$

$$x_{n-1}=r\sin\phi_1\sin\phi_2\sin\phi_3\ldots\sin\phi_{n-2}\cos\phi_{n-1},$$

$$x_n\ =r\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-2}\sin\phi_{n-1}$$

The new coordinates $r, \phi_1, \phi_2, \ldots\phi_{n-1}$ may be termed *spherical coordinates*. When these are varied one at a time while all the rest are left constant, the solution will trace out the following elements of way:

$$dr,\ rd\phi_1,\ r\sin\phi_1 d\phi_2,\ r\sin\phi_1\sin\phi_2 d\phi_3, \ldots,$$

$$r\sin\phi_1\sin\phi_2\ldots\sin\phi_{n-2}d\phi_{n-1},$$

whose cosines of direction form an orthogonal system; whence the product of these way-elements, viz.

$$r^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\ldots\sin^2\phi_{n-3}\sin\phi_{n-2}dr\,d\phi_1 d\phi_2\ldots d\phi_{n-1}$$

represents the element of totality, elsewhere expressed by $dx_1dx_2\ldots dx_n$. It is also clear that the element of the polyspheric continuum with radius 1, which in terms of the original coordinates might for instance be expressed in the form $\frac{1}{x_1}dx_2dx_3\ldots dx_n$ under the condition $x_1^2+x_2^2+\ldots+x_n^2=1$, is

Octavo, price 7s. 6d. each.

Cambridge Essays, 1857.

The Characteristics of English Criminal Law. By FITZJAMES STEPHEN, LL.B.
Modern Agriculture in Britain. By ANDREW STEUART, M.A., M.P.
Telegraphic Communication with India. By FRANCIS GISBORNE, M.A.
Porson. By H. R. LUARD, M.A.
Geology. By WILLIAM HOPKINS, M.A., F.R.S.
The Questions raised by the Mutiny. By CHARLES BUXTON, M.A., M.P.

Oxford Essays, 1857.

The Place of Homer in Education and in History. By the Right Honourable W. E. GLADSTONE, M.P., M.A.
Sicily. By M. E. GRANT DUFF, M.A.
Schemes of Christian Comprehension. By the Rev. H. B. WILSON, B.D.
Ancient Greece and Mediæval Italy. By E. A. FREEMAN, M.A.
The Burnett Prizes. By the Rev. BADEN POWELL, M.A., F.R.S.
The Jews of Europe in the Middle Ages. By J. H. BRIDGES, B.A.
Montaigne's Essays. By the Rev. R. W. CHURCH, M.A.
Characteristics of Thucydides. By W. I. SELLAR, M.A.

Cambridge Essays, 1856.

Roman Law and Legal Education. By H. J. S. MAINE, LL.D.
On English Ethnography. By J. W. DONALDSON, D.D.
Old Studies and New. By JOHN GROTE, B.D.
Taste for the Picturesque among the Greeks. By E. M. COPE, M.A.
The Apocryphal Gospels. By C. J. ELLICOTT, M.A.
Protestant Church and Religious Liberty in France. By W. H. WADDINGTON, M.A.
The Fly-Fisher and his Library. By H. R. FRANCIS, M.A.
The Text of Shakspeare. By CHARLES BADHAM, D.D.
Coleridge. By F. J. A. HORT, M.A.

Oxford Essays, 1856.

Comparative Mythology. By MAX MÜLLER, M.A.
The Growth of Laws and Usages of War. By MONTAGU BERNARD, B.C.L.
Raphael Drawings in the University Galleries. By G. BUTLER, M.A.
The Land System of Ireland. By W. O'CONNOR MORRIS, B.A.
National Education. By the Rev. FREDERICK TEMPLE, M.A.
Carlovingian Romance. By RICHARD JOHN KING, M.A.
Congreve's 'Roman Empire of the West.' By GOLDWIN SMITH, M.A.

Cambridge Essays, 1855.

The Life and Genius of Molière. By C. K. WATSON, M A.
The English Language in America. By C. A. BRISTED, B.A.
Notes on Modern Geography. By FRANCIS GALTON, M.A., F.G.S.
Limitations to Severity in War. By CHARLES BUXTON, M.P., M.A.
On the Transmutation of Matter. By G. D. LIVEING, M.A.
The Relations of Novels to Life. By FITZJAMES STEPHEN, B.A.
Prospects of the British Navy. By R. E. HUGHES, M.A.
Tennyson's Poems. By GEORGE BRIMLEY, M.A.
General Education and Classical Studies. By W. G. CLARK, M.A.

London: John W. Parker and Son, West Strand.

No. 8. **May, 1858.**

THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

EDITED BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH; AND

N. M. FERRERS, M.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE:

ASSISTED BY

G. G. STOKES, M.A., F.R.S.,

LUCASIAN PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE;

A. CAYLEY, M.A., F.R.S.,

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE; AND

M. HERMITE,

CORRESPONDING EDITOR IN PARIS.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστήμη πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

LONDON:

JOHN W. PARKER AND SON, WEST STRAND.

1858.

W. METCALFE, PRINTER, GREEN STREET, CAMBRIDGE.

PRICE FIVE SHILLINGS.

CONTENTS.

NOTICES TO CORRESPONDENTS.

We beg to acknowledge the Receipt of the following Papers, which will be inserted in the next Number:

MR. W. F. DONKIN, "On an Application of the Calculus of Operations to the Transformation of Trigonometric Series."

MR. R. B. HAYWARD, "Direct Demonstration of Jacobi's Canonical Formulæ for the Variation of Elements in a Disturbed Orbit."

MR. F. C. WACE, "On the Coincidence of the Two Rays in a Double Refracting Medium."

DR. SCHLAEFLI, "On the Multiple Integral $\int^n dx\,dy\ldots dz$, whose limits are

$p_1 = a_1x + b_1y +\ldots+ h_1z > 0,\ \ p_2 > 0,\ \ \ldots p_n > 0$, and $x^2 + y^2 +\ldots+ z^2 < 1$,"

will be completed in the next Number.

MR. CAYLEY, "On a Theorem Relating to Spherical Conics," "On the Wave Surface," and "Note on the Singular Solutions of Differential Equations."

We hope to continue the Rev. P. FROST'S "Planetary Theory" in the next Number, and expect an Article from the Rev. H. HOLDITCH.

Papers for the Journal and other communications may be addressed to the Editors under cover to Messrs. J. W. PARKER & SON, 445, *West Strand, London;* N. M. FERRERS, Esq., *Gonville and Caius College, Cambridge; or to the Printing Office, Green Street, Cambridge.*

The next Number will appear in October.

now expressed by

$$\sin^{n-2}\phi_1 \sin^{n-3}\phi_2 ... \sin^2\phi_{n-3} \sin\phi_{n-2} d\phi_1 d\phi_2 ... d\phi_{n-1},$$

which call ω for the moment. If then K be a portion of the totality bounded by the polyspheric continuum and by linear continua passing through the centre, and if S be the corresponding portion of the polyspheric continuum, we shall have

$$K = \int_{r=0}^{r=a} r^{n-1}dr \times \int^{n-1} \omega, \quad S = a^{n-1} \int^{n-1} \omega;$$

hence $$K = \frac{aS}{n}.$$

We are, of course, allowed to substitute S for K in our future reasonings, assuming also for shortness $a = 1$; and we propose the following terms for the different forms of S. If there be no linear boundaries, S may be called the *total polyspheric continuum*, having for its measure $2\pi^{\frac{n}{2}} : \Gamma\left(\frac{n}{2}\right)$; if there be any number greater than n of linear boundaries passing through the centre, we may say that S is a *polyspheric polyschemon*, and if that number be n, we may call it a *plagioschemon*, as having its $\frac{n(n-1)}{2}$ *arguments*, that is, the angles between every two of the linear boundaries, all arbitrary and so in general oblique; but in the particular case when the linear boundaries $p_1, p_2, ... p_n$ form a series (only invertible, but not cyclically permutable), in which each makes an arbitrary angle only with its immediate antecessor or successor, while it makes right angles with all the rest, we shall use the term *orthoschemon*. As for instance, the rectangular spherical triangle is of this last kind; for its sides form a series whose middle term, the hypothenuse, makes oblique angles with the extreme ones, while these, the extreme terms, include a right angle. The single parts, of which the whole enclosed figure consists, may be called *perischemons* primary, secondary, ... $(n-1)$-ary, with the meaning that a primary perischemon has some secondary ones for its own perischemons, properly so to be called, and so on. The ultimate and penultimate perischemons we may call respectively *vertices* and *sides*.

If $p_1, p_2, ... p_n$ be n linear polynomes without constant term and not connected by any identical relation, we are now to consider the plagioschemon S having $p_1 > 0$, $p_2 > 0$, ... $p_n > 0$ for its boundaries and the angles included by any two of them, (12), (13), ... ($1n$), (23), ... $\{(n-1)\,n\}$ for its *arguments*, so that for instance (12) denotes the angle between $p_1 > 0$, $p_2 > 0$. To avoid ambiguity we assume each argument to be positive and less than π. The coefficients in the polynomes p count as $n\,(n-1)$ arbitrary constants; but since the possibility of transforming the coordinates orthogonally implies $\frac{1}{2}n\,(n-1)$ disposable constants, we may subtract this number from the former, and so we bring the number of independent constants to be $\frac{1}{2}n\,(n-1)$. We may therefore assume the $\frac{1}{2}n\,(n-1)$ arguments of S to be the independent variables of which S is a function. Now we are at liberty to conceive the coordinate system such as to have $x_1, x_2, \ldots x_m$ vanishing together with $p_1, p_2, \ldots p_m$. If we then omit all terms in $x_1, x_2, \ldots x_m$ from the remaining polynomes $p_{m+1}, p_{m+2}, \ldots p_n$ and divide them by the positive square root of the sum of the squares of the coefficients of $x_{m+1}, x_{m+2}, \ldots x_n$, we shall indicate the polynomes thus altered by

$$p\,\{(123\ldots m),\ m+1\},\quad p\,\{(123\ldots m),\ m+2\},\quad \ldots p\,\{(123\ldots m),\ n\}.$$

When subjected to the condition of being all positive, these will determine that perischemon $S\,\{(123\ldots m)\}$ of the m-th descending order, which is the intersection of the polyspheric continuum and the linear one of the m conditions $p_1 = 0$, $p_2 = 0$, ... $p_m = 0$, yet bounded, of course, by the remaining polynomes $p_{m+1}, p_{m+2}, \ldots p_n$. Let, for instance, $\{(123\ldots m),\ (m+1)\,(m+2)\}$ denote the angle between

$$p\,\{(12\ldots m),\ m+1\} > 0 \quad\text{and}\quad p\,\{(12\ldots m),\ m+2\} > 0\,;$$

then all such angles will be the $\frac{1}{2}(n-m)\,(n-m-1)$ arguments of the perischemon $S\,\{(123\ldots m)\}$. There are, in the whole,* $\binom{n}{m}\binom{n-m}{2} = \binom{n}{2}\binom{n-2}{m}$ arguments of the order under consideration. At last come arcs of circle (the *sides*), as $S\,\{(345\ldots n)\}$, of which each is its own argument, that is, $S\,\{(345\ldots n)\} = \{(345\ldots n),\ 12\}$. Since the number of sides is

* For shortness we put

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \ldots.$$

$\frac{1}{2}n(n-1)$, we may also conceive the plagioschemon S to be a function of its sides. The aggregate number of all such angular quantities inherent to S, inclusively of the arguments properly so called and the sides, is

$$\sum_{m=0}^{m=n-2} \binom{n}{2}\binom{n-2}{m} = 2^{n-2}\binom{n}{2}.$$

The mode in which they are connected together and ultimately depend upon the arguments of S, may be thus stated. The angles between every three polynomes such as

$$p\{(12\ldots m), i\}, \quad p\{(12\ldots m), j\}, \quad p\{(12\ldots m), k\},$$

belonging to one perischemon $S\{(12\ldots m)\}$, namely

$$\{(12\ldots m), jk\}, \quad \{(12\ldots m), ik\}, \quad \{(12\ldots m), ij\}$$

are capable of being viewed as the angles of a spherical triangle having

$$\{(12\ldots mi), jk\}, \quad \{(12\ldots mj), ik\}, \quad \{(12\ldots mk), ij\}$$

for its sides, opposite in this order to the angles. The arguments of S being given, those of each perischemon may therefore be successively calculated by the help of such relations as

$$\cos\{(1), ik\} = \frac{\cos(ik) + \cos(1i)\cos(1k)}{\sin(1i)\sin(1k)}, \text{ etc.,}$$

$$\cos\{(12), ik\} = \frac{\cos\{(1), ik\} + \cos\{(1), 2i\}\cos\{(1), 2k\}}{\sin\{(1), 2i\}\sin\{(1), 2k\}}, \text{ etc.,}$$

etc.

From this sort of concatenation it is also plain, that *the supplements of the arguments will be the same functions of the supplements of the sides which the sides are of the arguments.*

Let us also attempt to represent the arguments of any perischemon immediately by those of the given plagioschemon S. It is plain that we must put

$$\rho p\{(12\ldots m), i\} = p_i + \lambda_1 p_1 + \lambda_2 p_2 + \ldots + \lambda_m p_m,$$

and determine the constant factors λ by the help of the condition of the new polynome being orthogonal to each of the polynomes $p_1, p_2, \ldots p_m$; whence they, together with ρ, must fulfil the equations

$$\begin{array}{l} 1-\rho^2 \quad -\lambda_1\cos(i1) \; -\lambda_2\cos(i2) \; -...-\lambda_m\cos(im)=0, \\ -\cos(1i)+\quad\lambda_1 \quad -\lambda_2\cos(12) \; -...-\lambda_m\cos(1m)=0, \\ -\cos(2i)-\lambda_1\cos(21)+\quad \lambda_2 \quad -...-\lambda_m\cos(2m)=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\cos(mi)-\lambda_1\cos(m1)-\lambda_2\cos(m2)-...+\quad\lambda_m \quad =0. \end{array}$$

Let i, ρ, λ change into κ, σ, μ; the sum of the products of the coefficients of the polynomes $p_i+\lambda_1 p_1+\lambda_2 p_2+...+\lambda_m p_m$ and $p_k+\mu_1 p_1+\mu_2 p_2+...+\mu_m p_m$ will then obviously be the same as if the latter polynome were replaced by p_k, since the former must be orthogonal to $p_1, p_2, ... p_m$. Consequently

$$\rho\sigma\cos\{(12...m), ik\}$$
$$=\cos(ik)+\lambda_1\cos(1k)+\lambda_2\cos(2k)+...+\lambda_m\cos(mk).$$

Hence, putting for shortness

$$\Delta\begin{pmatrix} i \\ k \end{pmatrix}123...m\Bigg) = \begin{vmatrix} -\cos(ik) & .-\cos(i1) & .-\cos(i2) & ....-\cos(im) \\ -\cos(1k) & . \quad 1 & .-\cos(12) & ....-\cos(1m) \\ -\cos(2k) & .-\cos(21) & . \quad 1 & ....-\cos(2m) \\ \cdots & \cdots & \cdots & \cdots \\ -\cos(mk) & .-\cos(m1) & .-\cos(m2) & .... \quad 1 \end{vmatrix}$$

and indicating this determinant by $\Delta(i123...m)$ when $k=i$ and therefore $-\cos(ik)=1$, we get

$$\rho\sigma\cos\{(12...m), ik\}\,.\,\Delta(123...m)+\Delta\left(\begin{matrix} i \\ k \end{matrix}123...m\right)=0,$$

$$-\rho^2\,.\,\Delta(123...m)+\Delta\;(i123...m)=0,$$

$$-\sigma^2\,.\,\Delta(123...m)+\Delta\;(k123...m)=0,$$

and therefore

$$\cos\{(123...m), ik\}=-\frac{\Delta\left(\begin{matrix} i \\ k \end{matrix}123...m\right)}{\sqrt{\{\Delta(i123...m)\}}\,.\sqrt{\{\Delta(k123...m)\}}},$$

where the square roots are meant to be positive. Hence also by a well-known property of determinants it follows that

$$\sin^2\{(123...m), ik\}=\frac{\Delta(ik123...m)\;\Delta(123...m)}{\Delta(i123...m)\;\Delta(k123...m)}.$$

After this preparation we are at liberty to throw the bounding polynomes into the forms

$$\left.\begin{array}{l} p_1 = \quad x_1, \\ p_2 = - x_1 \cos(12) + x_2 \sin(12), \\ p_3 = - x_1 \cos(13) - x_2 \sin(13) \cos\{(1), 23\} + x_3 \sin(13) \sin\{(1), 23\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_m = -\sum_{\lambda=1}^{\lambda=m} x_\lambda \prod_{\mu=1}^{\mu=\lambda-1} \sin[\{123...(\mu-1)\}, \mu m].\cos[\{123...(\lambda-1)\}, \lambda m], \\ \qquad\qquad [\text{for } m = 1, 2, 3, \dots n], \end{array}\right\} \quad \cdots\cdots\cdots\cdots(a),$$

where at the end $-\cos[\{123...(m-1\}, mm] = 1$. We then call the solution ($x_1 = 0$, $x_2 = 0$, ... $x_{n-1} = 0$, $x_n = 1$) the *summit* A of the plagioschemon S, and, accordingly, the perischemon $S\{(n)\}$ its *base*. Now let fall the normal AB from the summit A upon the linear continuum ($p_n = 0$) of the base, and put its length $AB = \sin h$. From the centre O draw a radius through B; its extremity C will lie upon the polyspherical base $S\{(n)\}$ and may be called the *foot* of the circular arc $AC = h$, which are we assume to be the *height* of the plagioschemon S. It is clear that $\sin h$ is nothing else than the value of p_n at the solution A and therefore equal to the coefficient of x_n in this polynome; hence

$$\sin h = \sin(1n) \sin\{(1), 2n\} \sin\{(12), 3n\} \sin\{(123), 4n\} \\ \dots \sin[\{12...(n-2)\}, (n-1)n],$$

where a permutation of the indices 1, 2, 3, ... $n-1$ is allowed. The coordinates of the foot C are

$$x_1 = \tan h \cos(1n), \; x_2 = \tan h \sin(1n) \cos\{(1), 2n\}, \; \dots, \; x_n = \cos h.$$

Lastly, let P be any solution situated in the polyspherical base $S\{(n)\}$ and call ϕ its arcual distance from the summit A, or the angle between the radii OA and OP. Now if we conceive a *space* (or linear treble continuum) to pass through the three radii OA, OC, OP, its intersection with the polyspherical continuum will be a spherical surface and ACP will be a right-angled triangle upon it, of which $AP = \phi$ is the hypothenuse. Let $\angle APC = \theta$, then $\sin h = \sin\phi \sin\theta$. Around P pick out of the base $S\{(n)\}$ an element σ of $n-2$ infinitely small dimensions and join all its solutions by circular arcs with the summit A; hence will arise a small part of the polyspherical continuum, reaching from A to P and having no finite extension other than this arc; and when we cut it at P normally to the tangent of the circle, the *measure of the transverse section will be* $\sigma \sin\theta$. But when the above mentioned part is produced up to the equation $x_n = 0$ (whose intersection with the polysphere call *equator* relating to A as *pole*), its section may be denoted by ω and may be called the *equatorial element* corresponding to σ.

On this mode of viewing rests the demonstration of the following lemma:

SECTION IV.

Lemma. If every element of a plagioschemon S be multiplied by the cosine of its arcual distance from one of the vertices, which may be regarded as the summit, then the sum of all these products will be the $(n-1)$-th part of the product of the measure of the base and of the sine of the height.

Dem. Let A be the summit, Q the solution at which the element lies, put the arc $AQ = \psi$ and the coordinates of Q

$$x_1 = \sin\psi . x_1', \; x_2 = \sin\psi . x_2', \; ..., \; x_{n-1} = \sin\psi . x'_{n-1}, \; x_n = \cos\psi;$$

the element of the polyspherical continuum will then be

$$\sin^{n-2}\psi \, d\psi . \omega,$$

where $\omega = \frac{1}{x_1'} dx_2' \, dx_3' \dots dx'_{n-1}$ [for $x_1'^2 + x_2'^2 + ... + x'^2_{n-1} = 1$] denotes the corresponding *equatorial* element. Now, to find the integral

$$\int^{n-1} \cos\psi \times \sin^{n-2}\psi \, d\psi . \omega$$

extended to the whole plagioschemon S, we first suppose $x_1', x_2', \dots x'_{n-1}$ constant and integrate from $\psi = 0$ to $\psi = \phi$ belonging to the base $(p_n = 0)$. We get

$$\frac{1}{n-1} \int^{n-2} \sin^{n-1}\phi . \omega.$$

But, as we saw before, the normal section at the base is

$$\sin^{n-2}\phi . \omega = \sigma \sin\theta = \frac{\sin h}{\sin\phi} \sigma,$$

whence the integral becomes

$$\frac{\sin h}{n-1} \int^{n-2} \sigma,$$

which was to be proved.

This lemma enables us to prove the following fundamental theorem:

Section V.

Theorem. "The first derivatives of the measure of a plagioschemon, taken with respect to its arguments, are the $(n-2)$-th parts of the measures of the corresponding secondary perischemons;" or

$$dS = \frac{1}{n-2}\Big\{S\{(12)\}\, d(12) + S\{(13)\}\, d(13) + \ldots + S\{(n-1)\,n\}\, d\{(n-1)\,n\}\Big\}.$$

Dem. In order to vary the single argument (12), let us vary only the polynome p_1; it becomes

$$(1+k_1)\,x_1 + k_2x_2 + k_3x_3 + \ldots + k_nx_n,$$

where $k_1, k_2, \ldots k_n$ denote infinitely small quantities. As the sum of the squares of the coefficients must remain equal to unity and the arguments (13), (14), ... $(1n)$ constant, there are $n-1$ conditions just sufficient to determine the $n-1$ ratios $k_1 : k_2 : \ldots : k_n$. The first condition

$$(1+k_1)^2 + k_2^2 + k_3^2 + \ldots + k_n^2 = 0$$

degenerates into $k_1 = 0$; for quantities of the second order may be neglected. But hereby the conditions altogether have become just such as if they should determine the coordinates of the vertex $S\{(1345\ldots n)\}$. If now we regard this as the summit of the perischemon $S\{(1)\}$ and accordingly $S\{(12)\}$ as its base, in acting then as if there were not the dimension x_1 and so applying the former results to an $(n-1)$-fold totality, we shall begin to look on the value of x_2 at the summit as $\sin h$, the sine of the height. Again, from the equation

$$-\cos\{(12) + d(12)\} = -\cos(12) + k_2 \sin(12)$$

which furnishes the angle $(12) + d(12)$ between the varied polynome $p_1' = x_1 + k_2x_2 + \ldots + k_nx_n$ and the unaltered one $p_2 = -x_1\cos(12) + x_2\sin(12)$, we infer $k_2 = d(12)$. Therefore $k_2, k_3, \ldots k_n$ have the same ratio to the homologous coordinates of the vertex $S\{(1345\ldots n)\}$ which $d(12)$ has to $\sin h$. Hence, if ϕ denote the arcual distance (from the last-mentioned vertex) of any solution $(0, x_2, x_3, \ldots x_n)$ lying upon the perischemon $S\{(1)\}$, we shall have

$$k_2x_2 + k_3x_3 + \ldots + k_nx_n = \cos\phi \frac{d(12)}{\sin h},$$

and therefore $p_1' = x_1 + \frac{\cos\phi}{\sin h} d(12)$. Now the boundaries of dS are $p_1 < 0$, $p_1' > 0$, or $0 < -x_1 < \frac{\cos\phi}{\sin h} d(12)$, while the remaining boundaries are the same as for S itself; but since x_1 is infinitely small, these remaining boundaries must be allowed to be taken as if x_1 were $= 0$, that is, we may, if we please, substitute for them the boundaries of $S\{(1)\}$, viz. $p\{(1), 2\}$, $p\{(1), 3\}$, ... $p\{(1), n\}$. Consequently we get dS by adding all the elements of $S\{(1)\}$, each multiplied by $\frac{\cos\phi}{\sin h} d(12)$; it then follows from the foregoing lemma, that

$$dS = \frac{d(12)}{\sin h} \times \frac{1}{n-2} \text{ base } S\{(12)\} . \sin h = \frac{1}{n-2} S\{(12)\} . d(12),$$

which was to be proved.

This form of the theorem has the inconvenience that it is not capable of being followed down to $n = 2$. To avoid this, we may return to the first proposed integral

$$K = \int^n dx_1 dx_2 \ldots dx_n (x_1^2 + x_2^2 + \ldots + x_n^2 < 1, p_1 > 0, p_2 > 0, \ldots p_n > 0),$$

which we saw to be $\frac{1}{n} S$, and in like manner we may introduce as a portion of the linear continuum $(p_1 = 0,\ p_2 = 0)$, having the polyspherical continuum and $p_3 > 0, p_4 > 0, \ldots p_n > 0$ for its boundaries, $K\{(12)\} = \frac{1}{n-2} S\{(12)\}$. The present theorem then assumes the more general form

$$dK = \frac{1}{n} \Big\{ K\{(12)\} d(12) + K\{(13)\} d(13) + \ldots + K\{(n-1)n\} d\{(n-1)n\} \Big\}.$$

When $n = 2$, this equation reduces itself to

$$dK = \tfrac{1}{2} K\{(12)\} d(12);$$

here K is a sector of a circle, (12) is its angle at the centre, and $K\{(12)\}$ is the measure of the centre in a totality of no dimension, which can be nothing but unity. That is to say: the differential of a sector of circle with radius 1 is half the differential of its angle at the centre. By integrating, we get $K=\frac{1}{2}(12)$.

When $n=3$, the general theorem gives us

$$dK=\tfrac{1}{3}\left[K\{(12)\}\,d(12)+K\{(13)\}\,d(13)+K\{(23)\}\,d(23)\right];$$

here K is a spherical pyramid, of which the spherical triangle S is the base; (12), (13), (23) are the angles of S; $K\{(12)\}$ is the length of the radius reaching to the vertex $S\{(12)\}$ and therefore $=1$. To determine the constant of integration, we cause K to vanish by assuming $p_1=p_2=-p_3$, which gives $(12)=\pi$, $(13)=(23)=0$. Thus we get

$$K=\tfrac{1}{3}\{(12)+(13)+(23)-\pi\},\ \text{ or }\ S=(12)+(13)+(23)-\pi.$$

It may be observed that each angle as a transcendental quantity implies one integration only, and that so the problem relative to space does not increase the difficulty at all, but reduces itself to that relative to the plane. The same thing occurs again in every totality whose number of dimensions is odd.

In the fourfold totality the plagioschemon $ABCD$ has the same configuration as a tetrahedron in space, save that its primary perischemons are spherical triangles. We may then say that its derivative taken with regard to the argument along the edge AB is equal to half this edge or side AB. If we conceive all the sides infinitely small, the present theorem leads, in fact, to a known expression of the volume of a tetrahedron in space.

In the fivefold totality the perischemon $S\{(12)\}$ becomes a spherical triangle, the area of which we already know; we have therefore

$$\frac{dS}{d(12)}=\tfrac{1}{3}[\{(12),34\}+\{(12),35\}+\{(12),45\}-\pi].$$

Ranging, in the complete expression of dS, the thirty terms similar to $\{(12),34\}\,d(12)$ according to the five combinations such as (1234), and observing that half the group alluded to is the complete differential of a tetraspheric plagioschemon with the arguments (12), (13), (14), (23), (24), (34), if then $S(1234)$ denote this plagioschemon and $S(12345)$ the proposed pentaspheric one, we shall have

$$dS(12345)=\tfrac{2}{3}d\{S(1234)+\text{etc.}\}-\frac{\pi}{3}d\{(12)+\text{etc.}\}.$$

In order to determine the constant of integration we assume all the arguments to be right angles, so that we obtain

$$S(12345) = \frac{\pi^2}{12} \text{ and } S(1234) = \frac{\pi^2}{8}.$$

It then follows that

$$S(12345) = \tfrac{2}{3}\{S(2345) + \text{etc.}\} - \frac{\pi}{3}\{(12) + \text{etc.}\} + \frac{4\pi^2}{3};$$

or there is reduction from five to four dimensions, just as in the case of space. We should at once follow up this observation, if we did not in the first place choose to disengage the expressions from Gamma-functions and powers of π.

Section VI.

We shall henceforth put

$$\underset{\begin{pmatrix} x_1^2 + x_2^2 + \ldots + x_n^2 < 1 \\ p_1 > 0,\, p_2 > 0,\, \ldots p_n > 0 \end{pmatrix}}{\int^n dx_1\, dx_2 \ldots dx_n} = f(123\ldots n) \times \underset{\begin{pmatrix} x_1^2 + x_2^2 + \ldots + x_n^2 < 1 \\ x_1 > 0,\, x_2 > 0,\, \ldots x_n > 0 \end{pmatrix}}{\int^n dx_1\, dx_2 \ldots dx_n},$$

or, what is the same thing,

$$K(123\ldots n) = \frac{1}{2^n} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} f(123\ldots n),$$

$$S(123\ldots n) = \frac{1}{2^{n-1}} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} f(123\ldots n),$$

and we shall call $f(123\ldots n)$ an n-spheric *plagioschematic function.* Every such function becomes unity, when all its arguments are $\frac{\pi}{2}$. We have, for instance,

$$f(12) = \frac{2}{\pi}(12), \quad f(123) = f(12) + f(13) + f(23) - 2,$$

$$f(12345) = f(2345) + \text{etc.} - 2\{f(12) + \text{etc.}\} + 16.$$

The fundamental theorem now assumes the form

$$df(123\ldots n) = f\{(12),\, 34\ldots n\}\, df(12) + \text{etc.}$$

Let us first suppose each of the m bounding polynomes $p_1, p_2, \ldots p_m$ to be orthogonal to each of the $n - m$ remaining ones $p_{m+1}, p_{m+2}, \ldots p_n$. Nothing prevents us from disposing of the coordinate system, so that in the former polynomes appear only the coordinates $x_1, x_2, \ldots x_m$. Hence if these should also appear in any one of the latter polynomes, we might, from the corresponding m conditions of orthogonality, infer that the determinant of the coefficients in the former polynomes vanishes, which on account of their independency must not be admitted. The latter polynomes therefore can only contain the remaining coordinates $x_{m+1}, x_{m+2}, \ldots x_n$. Let now

$$x_1^2 + x_2^2 + \ldots + x_m^2 = \cos^2\theta,$$

$$x_{m+1} = y_1 \sin\theta,\ x_{m+2} = y_2 \sin\theta,\ \ldots x_n = y_{n-m} \sin\theta,$$

and conceive the m former coordinates at first to be constant and the integration to be only performed with respect to the latter coordinates; then the corresponding limits, on having introduced the variables y, will plainly be the same which they before were in terms of $x_{m+1}, x_{m+2}, \ldots x_n$; but the limit $y_1^2 + y_2^2 + \ldots + y_{n-m}^2 < 1$ is to be added. Hence it follows

$$K(123\ldots n) = f\{(m+1)(m+2)\ldots n\}$$
$$\times \int^n \sin^{n-m}\theta . dx_1 dx_2 \ldots dx_m . dy_1 dy_2 \ldots dy_{n-m}$$
$$(p_1 > 0, p_2 > 0, \ldots p_m > 0, y_1 > 0, y_2 > 0, \ldots y_{n-m} > 0)$$

and by restoring the original coordinates

$$= f\{(m+1)(m+2)\ldots n\} \times \int^n dx_1 dx_2 \ldots dx_m dx_{m+1} dx_{m+2} \ldots dx_n \quad .$$
$$(p_1 > 0, \ldots p_m > 0, x_{m+1} > 0, \ldots x_n > 0)$$

If we here assume the latter coordinates first to be constant, we obtain in like manner as before

$$K(123\ldots n) = f\{(m+1)(m+2)\ldots n\} \times f(12\ldots m) \times \int^n dx_1 dx_2 \ldots dx_n \quad ;$$
$$\begin{pmatrix} x_1^2 + x_2^2 + \ldots + x_n^2 < 1 \\ x_1 > 0, \ldots x_n > 0 \end{pmatrix}$$

and hence, lastly,

$$f(123\ldots n) = f(12\ldots m) \times f\{(m+1)(m+2)\ldots n\};$$

or to express this in words, if the polynomes bounding a plagioschematic function form two sets such that all the polynomes of the one set are orthogonal to all those of the other set, this function will be the product of two lower functions corresponding each to one of the two sets.

Since, as is easily seen, $f(1) = 1$, the first polynome being orthogonal to all the rest causes $f(123\ldots n) = f(23\ldots n)$; and when the m first polynomes are orthogonal not only to all the rest, but also among themselves each to each, the equation $f(123 \ldots n) = f\{(m+1)(m+2) \ldots n\}$ will exist.

When two plagioschematic functions differ from one another simply by the opposition of signs of one bounding polynome, their sum will be double the function bounded only by all the remaining polynomes. To express this proposition in symbols, we write $f(p_1, p_2, \ldots p_n)$ instead of $f(123 \ldots n)$; so we have

$$f(p_1, p_2, \ldots p_n) + f(-p_1, p_2, p_3, \ldots p_n) = 2f(p_2, p_3, \ldots p_n).$$

For since, in the integral arising from the union of the two functions, the polynome p_1 has disappeared, the sum will be the same, what polynome soever be substituted for p_1, as for instance one which is orthogonal to all the rest. The consequence is obvious.

Section VII.

Reduction of a perissospheric plagioschemon.

In order to distinguish the two cases of an even and odd number of dimensions we shall use the terms *artiosphere* and *perissosphere;* and as to the reduction above announced of a perissospheric plagioschematic function into linear terms of artiospheric ones, we now state the following general theorem:

Theorem. "If f_{2n+1} be a plagioschematic function bounded by the polynomes $p_1, p_2, \ldots p_{2n+1}$, and if Σf_{2m} denote the sum of all $(2m)$-spheric functions bounded by any $2m$ of those polynomes (it is assumed that $f_0 = 1$), then

$$f_{2n+1} = \sum_{i=0}^{i=n} (-)^i a_i \Sigma f_{2n-2i} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

where the coefficients a are defined by the expansion

$$\tan x = \sum_{i=0}^{i=\infty} a_i \frac{x^{2i+1}}{1.2.3 \ldots (2i+1)} \;\ldots\ldots\ldots\ldots\; (2)."$$

Dem. If we differentiate the equation (1) with regard to any argument of f_{2n+1}, as for instance to (12), the last term $(-1)^n a_n$ on the right-hand side disappears, and we get

$$f_{2n-1}\{(12)\} = \sum_{i=0}^{i=n-1} (-)^i a_i \Sigma f_{2n-2i-2}\{(12)\},$$

a similar equation, wherein only the number $2n+1$ of dimensions is replaced by $2n-1$ and the bounding polynomes by

$p\{(12), 3\}$, $p\{(12), 4\}$, ... $p\{(12), n\}$. If then the truth of the theorem were admitted for the $(2n-1)$-sphere, we might infer (1) from integrating this latter equation (or rather the complete differential equation to be deduced from it), though having still to prove that the constant $(-1)^n a_n$ of integration has been duly determined. In fact, if we suppose all arguments of f_{2n+1} to be right angles and consider that the sum Σf_{2n-2i} counts for as many terms as $2n+1$ elements can be combined by $2n-2i$, then the equation (1) will become

$$1 = \sum_{i=0}^{i=n} (-)^i a_i \binom{2n+1}{2i+1},$$

or, divided by $1.2.3...(2n+1)$,

$$\sum_{i=0}^{i=n} \frac{(-)^{n-i}}{1.2.3...(2n-2i)} \cdot \frac{a_i}{1.2.3...(2i+1)} = \frac{(-)^n 1}{1.2.3...(2n+1)};$$

which same equation is also found by equating in the expansion of $\cos x \times \tan x = \sin x$ (where $\tan x$ is to be replaced by the series (2)) the coefficients of x^{2n+1} on both sides. The constant of integration therefore would be exactly determined, if the theorem hold good for $2n-1$ dimensions. Now, since $a_0 = 1$, $a_1 = 2$ and for the trisphere the equation $f_3 = \Sigma f_2 - 2$ is true, the theorem is generally proved.

Section VIII.

Dissection of a plagioschemon into orthoschemons.

We go on to a very important reduction which brings the number of arbitrary arguments of the considered n-spheric function from $\frac{1}{2}n(n-1)$ down to $n-1$.

When the polynomes $p_1, p_2, ... p_n$ bounding a plagioschemon S are *in this order* orthogonal each to all others but the antecessor and successor, so that only the $n-1$ arguments (12), (23), (34), (45), ... $\{(n-1)n\}$ may be oblique angles, we call S an *orthoschemon*, and our aim here is to show that every n-spheric plagioschemon can be dissected into $1.2.3...(n-1)$ orthoschemons, the arguments of which are connected with those of the given plagioschemon by trigonometrical relations.

To illustrate the conception of an orthoschemon, we lay down such a coordinate system that each of the bounding polynomes contains one more coordinate than its antecessor, which must be admitted as always possible. We then readily see the polynomes coming into these forms:

$$p_1 = x_1,$$
$$p_2 = -x_1\cos(12) + x_2\sin(12),$$
$$p_3 = \qquad -x_2\cos\{(1),23\} + x_3\sin\{(1),23\},$$
$$p_4 = \qquad\qquad -x_3\cos\{(12),34\} + x_4\sin\{(12),34\},$$

....

$$p_n = -x_{n-1}\cos[\{123\ldots(n-2)\},(n-1)\,n] + x_n\sin[\{123\ldots(n-2)\},(n-1)\,n].$$

The bounding polynomes and arguments of the perischemon $S\{(m)\}$ are

$$p\{(m),\,m-1\} = \frac{p_{m-1} + p_m\cos\{(m-1)\,m\}}{\sin\{(m-1)\,m\}},$$

$$p\{(m),\,m+1\} = \frac{p_{m+1} + p_m\cos\{m\,(m+1)\}}{\sin\{m\,(m+1)\}},$$

and for every index i different from $m-1$, m, $m+1$, we have

$$p\{(m),\,i\} = p_i;$$

and moreover

$$\cos\{(m),\,(m-2)\,(m-1)\} = \frac{\cos\{(m-2)\,(m-1)\}}{\sin\{(m-1)\,m\}},$$

$$\cos\{(m),\,(m+1)\,(m+2)\} = \frac{\cos\{(m+1)\,(m+2)\}}{\sin\{m\,(m+1)\}},$$

$$\cos\{(m),\,(m-1)\,(m+1)\} = \cot\{(m-1)\,m\}\cot\{m\,(m+1)\};$$

besides

$$\{(m),\,i\,(i+1)\} = \{i\,(i+1)\},\ \text{when } i=1,2,3,\ldots m-3,\,m+2,\ldots n-1;$$

and all the other arguments of $S\{(m)\}$ are right angles. Hence we may readily infer that all perischemons of an orthoschemon, whatever their number of dimensions be, are themselves also orthoschemons, and that the polynomes, by which each of them is bounded, follow the same order (though interrupted by gaps) as their indices in the original orthoschemon.*

The least number of orthoschemons, into which a spherical triangle can be divided, takes place when, from any vertex, a perpendicular arc is let fall upon the opposite side; I mean the two right-angled triangles. But in a more general manner we may also get six, that is 1.2.3, such triangles by drawing from any inner point three arcs perpendicular to the sides and other three arcs up to the vertices. To save the imagination from dealing with negative parts, we may suppose all the arguments of a given plagioschemon to be acute angles, and we may take an inner solution as a point of departure for the dissection. The analytical formulæ next suggested by this restrained mode of viewing would then nevertheless restore the somewhat impaired generality. In the fourfold totality we are helped by the idea of a tetrahedron where we let fall from an inner point perpendiculars upon each face and take their feet anew as points of departure for dividing each face in six right-angled triangles; then these twenty-four triangles will determine so many tetrahedrons, having that inner point for a common summit. One such corresponds, for instance, to the permutation (1234), if we let $S\{(1)\}$, $S\{(2)\}$, $S\{(3)\}$, $S\{(4)\}$ denote the faces of the original tetrahedron S; and we may point it out as follows. From the inner point A we let fall upon the face $S\{(1)\}$ a perpendicular whose foot let be $A\,(1)$; from this we draw a perpendicular to the edge $S\{(12)\}$, being the intersection of the faces $S\{(1)\}$ and $S\{(2)\}$, and call $A\,(12)$ its foot; from this we follow the same edge up to the vertex $S\{(123)\}$, which we may likewise consider as a foot $A\,(123)$. Then A, $A\,(1)$, $A\,(12)$, $A\,(123)$ are the vertices of the tetrahedron, or rather orthoschemon, which we were to point out as corresponding to the permutation (1234); its faces follow this order: the first coinciding with $S\{(1)\}$, the second passing through $S\{(12)\}$ and A, the third passing through $S\{(123)\}$, A and $A\,(1)$, the fourth passing through A, $A\,(1)$ and $A\,(12)$; the first stands perpendicular upon the third and fourth, and the second upon the fourth; wherefore the term orthoschemon applies here.

The order to be followed in the dissection of any plagioschemon being now sufficiently explained, we go on to analytical expositions. Let us assume one permutation, as for instance this $(123\ldots n)$, to be a rule on which we are to proceed, and let us therefore take, what must be allowed, an orthogonal coordinate system such that each of the polynomes $p_1, p_2, \ldots p_n$ contains one more coordinate than its antecessor, while the index of each additional coordinate agrees with that of the polynome (see (a) in Sec. 3.). This system is so arranged that we may immediately read off from it also the polynomes bounding the perischemons $S\{(1)\}$, $S\{(12)\}$, $S\{(123)\}$, $\ldots S\{123\ldots(n-2)\}$. Indicating the polynomes in the usual manner and for conformity those of S by $p(1), p(2), \ldots p(n)$ instead of $p_1, p_2, \ldots p_n$, we have by a sort of reversion

$$p\{(1), m\} = \frac{p(1)\cos(1m) + p(m)}{\sin(1m)},$$
$$[m = 2, 3, 4, \ldots n],$$

$$p\{(12), m\} = \frac{p\{(1), 2\}\cos\{(1), 2m\} + p\{(1), m\}}{\sin\{(1), 2m\}},$$
$$[m = 3, 4, \ldots n],$$

$$p\{(123), m\} = \frac{p\{(12), 3\}\cos\{(12), 3m\} + p\{(12), m\}}{\sin\{(12), 3m\}},$$
$$[m = 4, 5, \ldots n],$$

....

$$p[\{12\ldots(n-2)\}, m] = \frac{p[\{12\ldots(n-3)\}, n-2]\cos[\{12\ldots(n-3)\}, (n-2)m] + p[\{12\ldots(n-3)\}, m]}{\sin[\{12\ldots(n-3)\}, (n-2)m]},$$
$$[m = n-1, n],$$

and, lastly,

$$p[\{12\ldots(n-1)\}, n] = x_n = \frac{p[\{12\ldots(n-2)\}, n-1]\cos[\{12\ldots(n-2)\}, (n-1)n] + p[\{12\ldots(n-2)\}, n]}{\sin[\{12\ldots(n-2)\}, (n-1)n]}.$$

This inverse system gives us the coordinates of every solution

$$x_1 = p(1),\ x_2 = p\{(1), 2\},\ x_3 = p\{(12), 3\},\ x_4 = p\{(123), 4\}, \ldots,$$

when the polynomes $p(1), p(2), \ldots p(n)$, that is the *distances* from that solution to the linear continua bounding S, are known; it therefore enables us to represent these coordinates appropriate to the permutation $(123\ldots n)$ in a way independent of the accidental coordinate system. It is worth remark that the equation of the polysphere changes into

$$p\{(1)\}^2 + p\{(1), 2\}^2 + p\{(12), 3\}^2 + p\{(123), 4\}^2 + \ldots + p[\{12\ldots(n-1)\}, n]^2 = 1,$$

or, what is the same thing, into

$$\begin{vmatrix} 1 & p(1) & p(2) & p(3) & \ldots & p(n) \\ p(1) & 1 & -\cos(12) & -\cos(13) & \ldots & -\cos(1n) \\ p(2) & -\cos(21) & 1 & -\cos(23) & \ldots & -\cos(2n) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ p(n) & -\cos(n1) & -\cos(n2) & -\cos(n3) & \ldots & 1 \end{vmatrix} = 0.$$

An immediate proof of this equation results from considering that the determinant is nothing else than

$$\left\|\begin{array}{c|c} x_1, x_2, \dots x_n & x_1, x_2, \dots x_n \\ \text{coefficients in } p(1) & \text{coefficients in } p(1) \\ \text{“} \quad \text{“} \; p(2) & \text{“} \quad \text{“} \; p(2) \\ \dots\dots\dots & \dots\dots\dots \end{array}\right\|,$$

and that, since each rectangle here contains one horizontal line more than it contains vertical ones, the expression must identically vanish.

Now if we state the as yet moveable solution to be the common vertex A of all the orthoschemons which we have to cut out, then the letter p, as a sign of polynome belonging to an indeterminate solution, may be exchanged for a, as a sign of the polynome belonging to A, so that

$$a(1),\ a\{(1), 2\},\ a\{(12), 3\},\ \dots a[\{12\dots(n-1)\}, n]$$

will be the coordinates of A in the peculiar system above adopted. Those of the successive feet $A(1)$, $A(12)$, $A(123)$, ... will then readily result from this set by rejecting as many of its first terms as is indicated by the order of the foot and dividing all the rest by the square root of the sum of their squares. Again, let $q_1, q_2, \dots q_n$ denote the polynomes bounding the orthoschemon relative to the permutation $(123\dots n)$; then $q_1 = p(1)$; q_2 passes through $S\{(12)\}$ and A, and, apart from a constant factor, it may therefore be written

$$-a\{(1), 2\}\,p(1) + a(1)\,p\{(1), 2\};$$

in general, q_m passes through $S\{(12\dots m)\}$ and A, $A(1)$, $A(12)$, $A(123)$, ... $A\{12\dots(m-2)\}$, which condition is satisfied by

$$-a[\{12\dots(m-1)\}, m]\,p[\{12\dots(m-2)\}, m-1] \\ + a[\{12\dots(m-2)\}, m-1]\,p[\{12\dots(m-1)\}, m],$$

and lastly for q_n which passes through A, $A(1)$, $A(12)$, ... $A\{123\dots(n-2)\}$ we obtain a formula subject to the same law. It may be observed in passing, that the expression just written is identical with

$$a[\{12\dots(m-2)\}, m-1]\,p[\{12\dots(m-2)\}, m] \\ - a[\{12\dots(m-2)\}, m]\,p[\{12\dots(m-2)\}, m-1]$$

the whole divided by

$$\sin[\{12\dots(m-2)\}, (m-1)\,m],$$

and that therefore it remains the same, however the indices $1, 2, 3, \ldots m-2$ be permuted, but that it changes its sign by permuting $m-1$ and m. Hence we get an idea of the adjacency of different orthoschemons. As regards the sign of q_m, we must make that ($a(1)$, $a\{(1), 2\}$, $a\{(12), 3\}$, ... all being supposed positive) it become positive for the opposite vertex $A\{12\ldots(m-1)\}$. This being already the case, we have not to change the sign. Since $p[\{12\ldots(m-2)\}, m-1]$, $p[\{12\ldots(m-1)\}, m]$ themselves are the coordinates x_{m-1}, x_m in the peculiar system above employed, we may regard their coefficients (in the above expression = constant × q_m) as the base and perpendicular of a plane right-angled triangle, and we may accordingly put

$$\tan\beta_{m-1} = \frac{a[\{12\ldots(m-2)\}, m-1]}{a[\{12\ldots(m-1)\}, m]},$$

so as to have

$$q_m = -\cos\beta_{m-1}.p[\{12\ldots(m-2)\}, m-1] + \sin\beta_{m-1}.p[\{12\ldots(m-1)\}, m].$$

This agrees completely with the above mentioned mode of representing the polynomes bounding an orthoschemon. Now, if $\angle(12)$, $\angle(23)$, $\angle(34)$, ... $\angle\{(n-1)\,n\}$ denote the arguments of the orthoschemon under consideration, we have

$$\cos\angle(12) = \cos\beta_1,\ \cos\angle(23) = \sin\beta_1\cos\beta_2,\ \cos\angle(34) = \sin\beta_2\cos\beta_3,$$
$$\cos\angle(45) = \sin\beta_3\cos\beta_4,\ \ldots\ \cos\angle\{(n-1)\,n\} = \sin\beta_{n-2}\cos\beta_{n-1}.$$

Disregarding a constant factor, we may also write the equation $q_m = 0$ thus:

$$\left|\begin{matrix} -\cos(i1).-\cos(i2).-\cos(i3).\ldots-\cos\{i(m-2)\}.a(i).p(i) \\ [i = 1, 2, 3, \ldots m] \end{matrix}\right| = 0,$$

where $-\cos(ii)$ is to be replaced by 1. For it is plain first, that this determinant vanishes at every solution of the perischemon $S\{(12\ldots m)\}$ and at A; secondly, that it is orthogonal to $p(1), p(2), p(3), \ldots p(m-2)$, or, briefly, to $S[\{12\ldots(m-2)\}]$; consequently, that it passes not only through A, but also through $A(1)$, $A(12)$, $A(123)$, ... $A\{12\ldots(m-2)\}$. Also, this form shows that any permutation, whether of the indices $1, 2, \ldots m-2$, or of the indices $m+1, m+2, \ldots n$, has no influence here. We may hence readily infer the number of orthoschemons which have the unlimited linear continuum q_m for a common boundary.

What has hitherto been said, will, I think, clearly show how to find the arguments of all $1.2.3\ldots n$ orthoschemons, into which the given plagioschemon can be dissected from an arbitrary inner solution. Nor do I think it necessary to dwell on the simplifications which the general formulæ will undergo when applied to a vertex as a solution of departure for the dissection.

Section IX.

To reduce perissospherical orthoschemons to artiospherical ones.

At first sight this problem seems to be already solved together with that relating to plagioschemons; and for any small number of dimensions there is not much difficulty, it is true, in applying the former general formula to the more special problem here proposed. But in proceeding thus, it is scarcely possible to attain to a general law; and so we prefer undertaking to solve the present problem immediately without recourse to the former formula.

The nature of the subject in question gives rise to a preliminary remark. If $f(123...n)$ be an orthoschematic function, where the figures refer to the bounding polynomes, and if some of these polynomes be taken away, so that their significant order is here and there interrupted by gaps: then, in the new function formed by the remaining polynomes, the polynomes of each continuous series will be orthogonal to all those not contained in the same series, and therefore the function itself will break up into as many orthoschematic factors as there are continuous series between the gaps. For instance, if $i+1<m<n$, then

$$f\{123...i.m(m+1)...n\}=f(123...i).f\{m(m+1)...n\}.$$

In the following proposition only artiospheric factors can be employed.

Theorem. "Let f_{2n+1} be an orthoschematic function bounded by $2n+1$ polynomes given in their characteristic order, take away from this $2i+1$ polynomes in all possible ways, but so that each continuous series between two gaps should contain an even number of polynomes, and let Σf_{2n-2i} denote the sum of all the functions corresponding to these combinations (any such function being indecomposable or a product of functions, according as the series is continuous or interrupted): then the formula

$$f_{2n+1}=\sum_{i=0}^{i=n}\frac{(-)^i}{i+1}\binom{2i}{i}\Sigma f_{2n-2i}$$

holds good."

Dem. It may first be asked, in how many ways $2i+1$ terms can be struck out from the series $1, 2, 3, \dots 2n+1$ in such manner that an even number of terms is left in every continuous group. Beginning from the left hand we put the letters of the alphabet in their order under each *single* figure struck out and under each *pair* of the figures not struck out; thus we come to employ $n+i+1$ letters, of which those written under the figures struck out will form some combination of the $(2i+1)$-th class. The same combination cannot arise from another arrangement of the figures; and reciprocally given any combination whatever (to this class) of the letters, it is easy to find the corresponding arrangement of the figures, and there is one such arrangement only. Consequently, the number of such arrangements of the figures is that of the combinations of $n+i+1$ letters without repetition to the $(2i+1)$-th class. The sum Σf_{2n-2i} therefore contains $\binom{n+i+1}{2i+1}$ terms.

Now, for a preliminary trial of the constants in the asserted formula, we assume the $2n+1$ polynomes to be all orthogonal among themselves; each function f having then unity for its value, the question is whether the formula

$$1 = \sum_{i=0}^{i=n} \frac{(-)^i}{i+1} \binom{2i}{i} \binom{n+i+1}{2i+1}$$

be true. If h_n denote the sum on the right-hand side, we have

$$h_n - h_{n-1} = \sum_{i=0}^{i=n} \frac{(-)^i}{i+1} \binom{2i}{i} \binom{n+i}{2i} = \frac{1}{n} \Sigma (-)^i \binom{n+i}{i+1} \binom{n}{i}$$

$$= -\frac{1}{n} \Sigma \binom{-n}{i+1} \binom{n}{n-i} = -\frac{1}{n} \binom{0}{n+1} = 0;$$

hence $h_n = h_{n-1} = h_{n-2} = \dots = h_0 = 1$, which was to be proved.

In differentiating then the asserted formula with respect to any one of the arguments of f_{2n+1}, we shall meet with the same formula, but applying now to $2n-1$ dimensions, and in like manner for every other argument. If, therefore, the corresponding formula be true in the $(2n-1)$-fold totality, the equation obtained from the complete differentiation of the formula will be also true, and thence by integration the formula itself is shewn to be true; whence by integrating follows the truth of the original formula, since it already appears that the constant of integration is exact. To complete the demonstration it is only necessary to add that the equation

$$f(123) = f(23) + f(12) - 1$$

is satisfied.

We subjoin two more examples to be made use of hereafter:

$$f(12345) = f(2345) + f(1234) + f(12)\,f(45) - \{f(45) + f(34) + f(23) + f(12)\} + 2,$$

$$\begin{aligned} f(1234567) &= f(234567) + f(123456) + f(12)\,f(4567) + f(1234)\,f(67) \\ &- \{f(4567) + f(3456) + f(2345) + f(1234) + f(34)\,f(67) + f(23)\,f(67) \\ &+ f(23)\,f(56) + f(12)\,f(67) + f(12)\,f(56) + f(12)\,f(45)\} \\ &+ 2\,\{f(67) + f(56) + f(45) + f(34) + f(23) + f(12)\} - 5. \end{aligned}$$

(*To be continued*).

ON THE n^{th} CAUSTIC, BY REFLEXION FROM A CIRCLE.

By the Rev. HAMNET HOLDITCH, M.A., Senior Fellow of Gonville and Caius College, Cambridge.

WHEN a ray of light, issuing from a luminous point, is reflected from a circle, the locus of the intersection of the reflected ray with the next is the *first* Caustic: the ray however may be reflected a second time and will produce the *second* Caustic: and the n^{th} Caustic is that produced by a ray after n reflexions.

The following discussion will be similar in arrangement to that in the first volume of this *Journal*, and the necessary adoption of a general method will make the consideration of the first caustic more simple and distinct, from the retention of n in the operations; and the results formerly obtained for the *first*, will be interesting verifications of those now obtained for the n^{th} caustic.

Three or four caustics may be seen in the reflections of a mourning ring on a sheet of paper, the second a very beautiful object; they were pointed out to me by Professor Miller, and led to this investigation.

The luminous point being L (fig. 1), O the centre of the reflector, and $LBCD$ the course of a ray; let $OL = a$, $OA = b$, $\angle ALB = \theta$, and OBL the angle of incidence $= \alpha$: x and y the coordinates of P, a point in the n^{th} reflected ray, measured from the centre.

Therefore
$$AOB = \theta - \alpha,$$
$$BOC = \pi - 2\alpha,$$

when CD is the second reflected ray, but if it be the n^{th},
$$BOC = (n-1)(\pi - 2\alpha);$$

and therefore
$$\angle AOD = (2n-1)\tfrac{1}{2}\pi - (2n\alpha - \theta),$$
$$\sin AOD = \sin(2n-1)\tfrac{1}{2}\pi \cos(2n\alpha - \theta),$$
and
$$\cos AOD = \sin(2n-1)\tfrac{1}{2}\pi \sin(2n\alpha - \theta).$$

Let $2n\alpha - \theta = P$; then, as $\sin(2n-1)\tfrac{1}{2}\pi = -(-1)^n$,
$$\sin AOD = -(-1)^n \cos P,$$
$$\cos AOD = -(-1)^n \sin P.$$

Also $OM + PS = OD$, or $x \cos AOD + y \sin AOD = b \sin\alpha$; therefore $x \sin P + y \cos P = -(-1)^n b \sin\alpha$ is the equation to the n^{th} reflected ray, which might also have been found from knowing the coordinates of its two ends.

If this equation be differentiated on the supposition of x and y being constant, x and y will be the coordinates of a point in the caustic; but it will be well to establish a few preliminary formulæ for reference, and find the values of two or three quantities which will frequently occur.

Let then the above equation, which is that of the tangent, be put under the form

$$x \sin P + y \cos P = A \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and differentiating, while x and y remain constant,

$$x \cos P - y \sin P = \frac{dA}{dP} = B \ldots\ldots\ldots\ldots (2);$$

therefore, also
$$\frac{dx}{dP}\sin P + \frac{dy}{dP}\cos P = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

by subtracting (2) from the complete differentiation of (1). Differentiating (2),

$$\frac{dx}{dP}\cos P - \frac{dy}{dP}\sin P = x \sin P + y \cos P + \frac{dB}{dP} = A + \frac{dB}{dP} = C \ldots (4),$$

$$\frac{d^2x}{dP^2}\sin P+\frac{d^2y}{dP^2}\cos P=-\left(\frac{dx}{dP}\cos P-\frac{dy}{dP}\sin P\right) \text{ from (3)},$$

or $$\frac{d^2x}{dP^2}\sin P+\frac{d^2y}{dP^2}\cos P=-C \quad\ldots\ldots\ldots\ldots(5),$$

and differentiating (4),

$$\frac{d^2x}{dP^2}\cos P-\frac{d^2y}{dP^2}\sin P=\frac{dx}{dP}\sin P+\frac{dy}{dP}\cos P+\frac{dC}{dP}$$

$$=\frac{dC}{dP}=D \quad\ldots\ldots\ldots\ldots(6).$$

From (1) and (2)

$$\left.\begin{array}{l} x=A\sin P+B\cos P \\ y=A\cos P-B\sin P\end{array}\right\}\quad\ldots\ldots\ldots\ldots(7),$$

$$\left.\begin{array}{l} \dfrac{dx}{dP}=\ \ C\cos P \\ \dfrac{dy}{dP}=-C\sin P\end{array}\right\}\quad\ldots\ldots\ldots\ldots(8),$$

$$\left.\begin{array}{l} \dfrac{d^2x}{dP^2}=-C\sin P+D\cos P \\ \dfrac{d^2y}{dP^2}=-C\cos P-D\sin P\end{array}\right\}\quad\ldots\ldots\ldots\ldots(9).$$

To find the quantities B and C; since

$$b\sin\alpha=a\sin\theta,$$

$$b\cos\alpha d\alpha=a\cos\theta d\theta,$$

and as $P=2n\alpha-\theta$, $dP=2nd\alpha-d\theta$

$$=2nd\alpha-\frac{b\cos\alpha d\alpha}{a\cos\theta}=\frac{2na\cos\theta-b\cos\alpha}{a\cos\theta}d\alpha;$$

therefore

$$B=\frac{dA}{dP}=-b\frac{\cos\alpha d\alpha}{dP}(-1)^n=-\frac{ab\cos\alpha\cos\theta(-1)^n}{2na\cos\theta-b\cos\alpha}\ \ldots\ldots(10).$$

Again, $$(-1)^n dB=\frac{ab(\cos\theta\sin\alpha d\alpha+\cos\alpha\sin\theta d\theta)}{2na\cos\theta-b\cos\alpha}$$

$$-\frac{ba\cos\alpha\cos\theta(2na\sin\theta d\theta-b\sin\alpha d\alpha)}{(\quad)^2},$$

but $\dfrac{d\alpha}{dP}=\dfrac{a\cos\theta}{2na\cos\theta-b\cos\alpha}$ and $\dfrac{d\theta}{dP}=\dfrac{b\cos\alpha}{2na\cos\theta-b\cos\alpha}$;

therefore

$$(-1)^n \frac{dB}{dP} = \frac{ab\,(a \cos^2\theta \sin\alpha + b \cos^2\alpha \sin\theta)}{(\quad)^2}$$

$$- \frac{ab \cos\alpha \cos\theta\,(2nab \sin\theta \cos\alpha - ab \sin\alpha \cos\theta)}{(\quad)^3}$$

$$= \frac{a^3 \cos^2\theta \sin\theta + ab^2 \cos^2\alpha \sin\theta}{(\quad)^2}$$

$$- \frac{a^2b \cos\alpha \cos\theta \sin\theta\,(2nb \cos\alpha - a \cos\theta)}{(\quad)^3}$$

$$= \frac{a \sin\theta}{(\quad)^3} \{(a^2 \cos^2\theta + b^2 \cos^2\alpha)\,(2na \cos\theta - b \cos\alpha)$$

$$- ab \cos\alpha \cos\theta\,(2nb \cos\alpha - a \cos\theta)\};$$

$$\therefore \quad \frac{dB}{dP} = \frac{a \sin\theta\,(-1)^n}{(2na \cos\theta - b \cos\alpha)^3}\,(2na^3 \cos^3\theta - b^3 \cos^3\alpha) \ldots\ldots (11).$$

Also, $\quad A = -(-1)^n b \sin\alpha = -(-1)^n a \sin\theta;$

therefore $C = A + \dfrac{dB}{dP}$

$$= \frac{a \sin\theta\,(-1)^n}{(\quad)^3}\,(2na^3 \cos^3\theta - b^3 \cos^3\alpha - \overline{2na \cos\theta - b \cos\alpha}|^3)$$

$$= \frac{a \sin\theta\,(-1)^n}{(\quad)^3}\,(2na^3 \cos^3\theta - 8n^3a^3 \cos^3\theta$$

$$+ 12n^2a^2b \cos^2\theta \cos\alpha - 6nab^2 \cos\theta \cos^2\alpha),$$

or $$C = -\frac{na^2 \sin 2\theta\,(-1)^n}{(2na \cos\theta - b \cos\alpha)^3} \{(n^2-1)\,a^2 \cos^2\theta + \overline{3na \cos\theta - b \cos\alpha}|^2\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

where the bracket is a *positive* quantity.

The equations (7), thus become

$$\left.\begin{aligned} x &= -(-1)^n a \sin\theta \left(\sin P + \frac{\cos P}{2n \tan\alpha - \tan\theta}\right) \\ y &= \quad (-1)^n a \sin\theta \left(\frac{\sin P}{2n \tan\alpha - \tan\theta} - \cos P\right) \end{aligned}\right\} \ldots (13),$$

from which the curve may be traced; for if any value of θ be assumed, $\sin\alpha = \dfrac{a \sin\theta}{b}$ is known, and therefore $P = 2n\alpha - \theta$, and therefore x and y are known.

An easier way of tracing would be by putting

$$2n \tan\alpha - \tan\theta = \tan Q,$$

for then

$$r^2 = x^2 + y^2 = a^2 \sin^2\theta \left(1 + \frac{1}{\tan^2 Q}\right) = \frac{a^2 \sin^2\theta}{\sin^2 Q} \quad \text{or} \quad r = \frac{a \sin\theta}{\sin Q};$$

and if ϕ be the angle the radius vector makes with the axis of x,

$$\tan\phi = \frac{y}{x} = -\frac{\sin P - \cos P \tan Q}{\cos P + \sin P \tan Q} = \tan(Q - P),$$

or

$$\phi = Q - P,$$

and a table for values of r and ϕ may thus be made.

The curve may however be sufficiently traced from remarkable points hereafter found.

On the Asymptotes.

These take place when the coordinates of the curve are infinite, or when (13) $2n \tan\alpha - \tan\theta = 0$; which, combined with $b \sin\alpha = a \sin\theta$, give

$$\left.\begin{aligned} \sin\alpha &= \frac{\sqrt{(4n^2a^2 - b^2)}}{b\sqrt{(4n^2 - 1)}} \\ \sin\theta &= \frac{\sqrt{(4n^2a^2 - b^2)}}{a\sqrt{(4n^2 - 1)}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(14)},$$

which substituted in (1) the equation to the tangent, that equation becomes the equation to the asymptotes, agreeing with first caustic, number (14).

Hence there are asymptotes only when $b > a$ and $< 2na$.

Hence also when there are asymptotes to the first caustic, there are asymptotes to all the caustics.

Since $\tan\theta = 2n \tan\alpha$, and therefore positive, $\theta < \frac{1}{2}\pi$, and there is no ambiguity of sign, and therefore only *one* asymptote.

If there be no asymptote to the first caustic, the n^{th} and succeeding caustics will have asymptotes, when $n > \dfrac{b}{2a}$.

The perpendicular from the centre on the asymptote

$$= \sqrt{\left(\frac{4n^2a^2 - b^2}{4n^2 - 1}\right)}.$$

Intersections of n$^{\text{th}}$ *Caustic with the Reflector.*

These will take place, when $x^2+y^2=b^2$,

or $$1=\sin^2\alpha\left\{1+\frac{1}{(2n\tan\alpha-\tan\theta)^2}\right\} \text{ from (4)},$$

$$\cos^2\alpha=\frac{\sin^2\alpha}{(2n\tan\alpha-\tan\theta)^2},$$

$$2n\tan\alpha-\tan\theta=\mp\tan\alpha;$$

therefore $$\tan\theta=(2n\pm1)\tan\alpha;$$

therefore $$\theta \text{ is always } < \tfrac{1}{2}\pi \quad\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Hence *every two successive caustics intersect each other in a point in the reflector.*

From (15), combined with $b\sin\alpha=a\sin\theta$, we derive

$$\left.\begin{aligned}\sin^2\theta&=\frac{a^2\,\overline{2n\pm1}\,|^2-b^2}{4n(n\pm1)a^2}\\ \sin^2\alpha&=\frac{a^2\,\overline{2n\pm1}\,|^2-b^2}{4n(n\pm1)b^2}\\ \cos^2\theta&=\frac{b^2-a^2}{4n(n\pm1)a^2}\\ \cos^2\alpha&=\frac{(b^2-a^2)\,\overline{2n\pm1}\,|^2}{4n(n\pm1)b^2}\end{aligned}\right\}\ldots\ldots\ldots\ldots (16);$$

and therefore the values of x and y are known from (13), but they may be simplified; for, since

$$2n\tan\alpha-\tan\theta=\mp\tan\alpha,$$

$$x=-(-1)^nb\sin\alpha\left(\sin P\mp\frac{\cos P}{\tan\alpha}\right)=\pm(-1)^nb(\mp\sin P\sin\alpha+\cos P\cos\alpha)$$
$$=\pm(-1)^nb\cos(P\pm\alpha),$$

$$y=(-1)^nb\sin\alpha\left(\mp\frac{\sin P}{\tan\alpha}-\cos P\right)=\mp(-1)^nb(\sin P\cos\alpha\pm\cos P\sin\alpha)$$
$$=\mp(-1)^nb\sin(P\pm\alpha),$$

or $$\left.\begin{aligned}x&=\pm(-1)^nb\cos\{(n\pm1)\alpha-\theta\}\\ y&=\mp(-1)^nb\sin\alpha\{(n\pm1)\alpha-\theta\}\end{aligned}\right\}\ldots\ldots\ldots (17)$$

are the coordinates of the intersections of the n^{th} caustic with the reflector, where the upper or lower signs must be taken throughout.

The intersections are impossible if $\sin^2\theta > 1$, or (16) $a > b$, and also if $\sin^2\theta$ be negative, or (16) $b > a\,(2n \pm 1)$.

Generally, there are *two* points of intersection; but only *one* in the first caustic, for the lower sign would make $\sin\theta$ infinite; and only *one* in the n^{th} caustic, if $b > (2n-1)\,a$,

and $< (2n+1)\,a$.

If $b = a$, or the luminous point be in the reflector, it touches the caustic at that point.

It is easily seen that (17) contains the coordinates of the extremities of the n^{th} reflected ray, and if this ray be that which *produces* the first intersection of the caustic with the reflector, its other extremity will be the second intersection, though this ray does not *produce* the latter, a fact for which it is not easy to assign any *à priori* reason.

If the luminous point be without the reflector, the n[th] *caustic is wholly within.*

For (13) $$\frac{x^2+y^2-b^2}{b^2} = -\cos^2\alpha + \frac{\sin^2\alpha}{(2n\tan\alpha - \tan\theta)^2}$$
is negative if
$$(2n\tan\alpha - \tan\theta)^2 > \tan^2\alpha;$$
therefore if $$(2n-1)\tan\alpha > \tan\theta,$$
or $$\overline{2n-1}|^2 \tan^2\alpha > \frac{b^2\tan^2\alpha}{a^2 + (a^2-b^2)\tan^2\alpha},$$
$$\left\{\text{for } \tan^2\theta = \frac{b^2\tan^2\alpha}{a^2+(a^2-b^2)\tan^2\alpha},\ \text{from } b\sin\alpha = a\sin\theta\right\},$$
or $$\overline{2n-1}|^2 > \frac{b^2}{a^2+(a^2-b^2)\tan^2\alpha},$$
which is so, for the first side is not < 1 and the second *is*, unless $b = a$ and $\tan\alpha = 0$.

If $\alpha = \frac{1}{2}\pi$, $\sin\theta = \dfrac{b}{a}$, and $x^2 + y^2 = b^2$, which shew that if a tangent be drawn from the luminous point to the reflector, the point touched is a point in the n^{th} caustic.

Hence all the caustics touch each other and the reflector at that point.

Cusps.

When there are cusps, $\dfrac{dy}{dx}$ is a vanishing fraction; and therefore from (8), $C = 0$; and C can only vanish when $\sin 2\theta = 0$, or in the three cases, when $\theta = 0$, π, or $\frac{1}{2}\pi$.

Case (1). If $\theta=0$, $\alpha=0$, and $P=2n\alpha-\theta=0$; and therefore from (13)

$$x=-(-1)^n\frac{b\sin\alpha\cos P}{2n\tan\alpha-\tan\theta}=-(-1)^n\frac{ab\cos\alpha\cos\theta\cos P}{2na\cos\theta-b\cos\alpha}$$

$$=-(-1)^n\frac{ab}{2na-b}.$$

Case (2). If $\theta=\pi$, $\alpha=0$, and $P=-\pi$, then $x=(-1)^n\dfrac{ab}{2na+b}$,

or
$$x=\mp(-1)^n\frac{ab}{2na\mp b}\quad\text{.................}\quad(18)$$

are the coordinates of the two cusps in the axis of x, corresponding to $\theta=0$ and $\theta=\pi$.

One cusp fails when $a=b$; if the light be outside, in both cases $\theta=\pi$, and the cusps correspond to $\alpha=\pi$, and $\alpha=0$; and the expressions are the same as before.

Case (3). If $\theta=\frac{1}{2}\pi$, $\sin P=\sin(2n\alpha-\frac{1}{2}\pi)=-\cos 2n\alpha$,

$$\cos P=\sin 2n\alpha,$$

and
$$\left.\begin{aligned}x&=+(-1)^n a\cos 2n\alpha\\ y&=-(-1)^n a\sin 2n\alpha\end{aligned}\right\}\quad\text{................}\quad(19)$$

are the coordinates of the cusp outside the axis of x, α being determined from $a=b\sin\alpha$.

This cusp only exists when the light is within, for $\sin\alpha=\dfrac{a}{b}$; and as $x^2+y^2=a^2$, *it* and the light are equally distant from the centre; it is evidently also at the foot of the perpendicular from the centre upon the n^{th} reflected ray, and its direction therefore perpendicular to the radius vector.

Maxima and Minima of x *and* y.

Those of y take place when $\dfrac{dy}{dx}$, or

$$-\tan P=0;\ \text{therefore}\ \sin(2n\alpha-\theta)=0,$$

and $2n\alpha-\theta=m\pi$; therefore $\sin\theta=\sin 2n\alpha\cos m\pi=\dfrac{b}{a}\sin\alpha$,

or
$$b\sin\alpha=+(-1)^m a\sin 2n\alpha\quad\text{...........}\quad(20),$$

from which α may be determined; therefore from (13), $y=(-1)^{m+n-1}b\sin\alpha$ is a *maximum or minimum value of* y, its position being known from $x = (-1)^{m+n-1} \dfrac{b \sin\alpha}{2n \tan\alpha - \tan 2n\alpha}$, and the several values of α must be found from the equations (20), viz.

$$b \sin\alpha \mp a \sin 2n\alpha = 0.$$

The value determined will be a *maximum or minimum*, according as $y \dfrac{d^2y}{dP^2}$ is $\mp$, or (7 and 9) as $A.C$ is $\pm$, or $\dfrac{\sin\alpha \sin 2\theta}{2n\alpha \cos\theta - b\cos\alpha}$; or according as $2n \tan\alpha - \tan 2n\alpha$ is positive or negative.

We have for the *maxima or minima values* of x, $\dfrac{dx}{dy} = 0$,

or $\tan P = \infty$; and therefore $2n\alpha - \theta = (2m+1)\frac{1}{2}\pi$;

therefore $\sin\theta = -\cos 2n\alpha \sin(2m+1)\frac{1}{2}\pi = \dfrac{b}{a}\sin\alpha$,

or $$b \sin\alpha = -(-1)^m a \cos 2n\alpha \quad\text{............ (21)},$$

whence the values of α corresponding to the two suppositions $m = 0$ and $m = 1$.

Therefore from (13), $x = (-1)^{m+n-1} b \sin\alpha$ is the value of a maximum or minimum of x, its position being determined from $y = (-1)^{m+n} \dfrac{b \sin\alpha}{2n \tan\alpha + \cot 2n\alpha}$.

The above value of x will be a maximum or minimum, according as $\dfrac{xd^2x}{dP^2}$ is $\mp$ or as $A.C$, or according as

$$2n \tan\alpha + \cot 2n\alpha$$

is positive or negative.

In the first caustic, $2n \tan\alpha + \cot 2n\alpha = \dfrac{1 + 3\tan^2\alpha}{2\tan\alpha}$, indicating a maximum, as shewn in a former communication.

Radius of Curvature.

Since $$\frac{dy}{dx} = -\tan P, \quad \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = -\frac{1}{\cos^3 P},$$

$$\frac{dx}{dP} = C \cos P \text{ from (8)},$$

$$\frac{1}{dP} d\left(\frac{dy}{dx}\right) = -\frac{1}{\cos^2 P},$$

and the product of the first two, divided by the third $= C$, or radius of curvature

$$= -\frac{na^2 \sin 2\theta \, (-1)^n}{(2na\cos\theta - b\cos\alpha)^3} \{(n^2-1)\, a^2\cos^2\theta + 3\,(na\cos\theta - b\cos\alpha)^2\} \quad \ldots\ldots\ldots\ldots\ldots(22).$$

There is no point of contrary flexure; for it has been shewn that the values which make the denominator vanish, belong to the asymptote.

The radius vanishes at the cusps.

When the light is external; at the point where the circle touches the caustic, $b = a\sin\theta$ and $\cos\theta$ is negative; and therefore radius $= \frac{4n^2-1}{4n^2}\, b$, and is independent of the position of the light.

If the light be in the circumference, $\theta = \alpha = \frac{1}{2}\pi$, but as $\frac{1}{2}\pi$ is the greatest limit of α and the least of θ, $\cos\alpha = -\cos\theta$; and therefore radius

$$= 2na\, \frac{n^2 - 1 + 3(n+1)^2}{(2n+1)^3} = \frac{4n(n+1)\, b}{(2n+1)^2} = b\left(1 - \frac{1}{(2n+1)^2}\right),$$

or less than that of the circle, as it must be.

At the intersections of the caustic and reflector

$$(2n \pm 1)\, a\cos\theta = b\cos\alpha \ (16);$$

and therefore radius

$$= 2na\sin\theta\, \{n^2 - 1 + 3\,(n\pm 1)^2\} = 4n\,(n\pm 1)\,(2n\pm 1)\, a\sin\theta$$
$$= 2\,(2n\pm 1)\sqrt{\{n\,(n\pm 1)\}}\sqrt{\{(2n\pm 1)\, a^2 - b^2\}}.$$

Hence two successive caustics not only *intersect* one another in the reflector (15), but have the same radius of curvature there.

There is a point *equidistant with the outer cusp and light* from the centre which admits of simple determination, and so is useful in tracing the curve, and corresponds to $n\tan\alpha = \tan\theta$, for then $2n\tan\alpha - \tan\theta = \tan\theta$; and therefore

$$x = -(-1)^n b\sin\alpha\left(\sin P + \cos P\,\frac{\cos\theta}{\sin\theta}\right),$$

$$y = \ \ (-1)^n b\sin\alpha\left(\frac{\sin P\cos\theta}{\sin\theta} - \cos P\right),$$

or

$$\left.\begin{aligned} x &= (-1)^n a\cos(2n\alpha - 2\theta), \\ y &= (-1)^n a\sin(2n\alpha - 2\theta) \end{aligned}\right\} \ \ldots\ldots\ldots\ldots \ (23),$$

where $$\left.\begin{aligned}\cos^2\theta &= \frac{b^2-a^2}{(n^2-1)\,a^2} \\ \text{and} \qquad \sin^2\theta &= \frac{n^2a^2-b^2}{(n^2-1)\,a^2}\end{aligned}\right\},$$

from the equation $b\sin\alpha = a\sin\theta$ and the one above, or $b > a$, $na > b$, and $n > 1$, which shew that the light must be within the reflector and that the first caustic does not possess this point.

The radius of curvature here $= \dfrac{na^2 \sin 2\theta}{n^3a^3\cos^3\theta}(n^2-1)\,a^2\cos^2\theta$

$$= \frac{2\sqrt{(n^2-1)}\sqrt{(n^2a^2-b^2)}}{n^2}.$$

On the Incipient form of the n[th] *Caustic.*

If the luminous point be in the centre of the reflector, all the n^{th} reflected rays will pass through the centre; let it be removed to an infinitesimal distance a, and the caustic will then spring into existence.

Since $\qquad b\sin\alpha = a\sin\theta, \quad \alpha = \dfrac{a\sin\theta}{b}$;

and on referring to (13)

$$\sin P = 2n\alpha\cos\theta - \sin\theta = \frac{2na\sin\theta\cos\theta}{b} - \sin\theta,$$

$$\cos P = \cos\theta + 2n\alpha\sin\theta = \cos\theta + \frac{2na\sin^2\theta}{b},$$

$$2n\tan\alpha - \tan\theta = \frac{2na\sin\theta}{b} - \tan\theta,$$

and expanding to the first power of a, we have

$$\frac{\cos P}{2n\tan\alpha - \tan\theta} = -\frac{\cos^2\theta}{\sin\theta} - \frac{2na\cos\theta}{b\sin\theta},$$

$$\frac{\sin P}{2n\tan\alpha - \tan\theta} = \cos\theta,$$

therefore $$\left.\begin{aligned} x &= (-1)^n\left(a + \frac{2na^2}{b}\cos^3\theta\right) \\ \text{and} \qquad y &= (-1)^n\frac{2na^2}{b}\sin^3\theta\end{aligned}\right\}.$$

Let $$x+(-1)^n a = x_1,$$

therefore $$x_1^{\frac{2}{3}} + y^{\frac{2}{3}} = \left(\frac{2na^2}{b}\right)^{\frac{2}{3}},$$

the equation to the *incipient form.*

All are similar and their linear dimensions, as 1, 2, 3......

The *luminous point is the centre of all the even ones*, and a point at an equal distance from the centre of the reflector on the opposite side, is the centre of all the odd ones (fig. 2).

When the light is at a small finite distance, the *even* caustics group round the light, and the odd ones round a similar point on the opposite side, the outside cusps being all in a circle through the luminous point; the successive cusps (19) are equidistant from each other, by the chord of an arc whose sine is $\frac{a}{b}$ and the directions of the cusps are all tangents to the circle. The first twelve caustics are represented in fig. 3, their tails getting gradually longer, until the twenty-fifth, which has an asymptote.

If the light move away from the centre, and $a=5$ and $b=6$, as in the first caustic, in order that they may be compared together, the *second* and *third* caustics will be represented in figs. 4 and 5.

As the light still moves away; on reaching the reflector, the asymptote disappears, and the curve is then known to be an epicycloid; and afterwards the curve is wholly within, as in fig. 6, where the luminous point is at a distance of one diameter from the centre of the reflector; and finally, when the rays are parallel, the ultimate form is again an epicycloid.

Length of the n^th^ Caustic.

The length of the ray (fig. 1) $= LB + (n-1)BC + CP$,

$$PD = MN - NS = x \sin AOD - y \cos AOD;$$

therefore since $$\sin AOD = -(-1)^n \cos P,$$

$$\cos AOD = -(-1)^n \sin P,$$

$$PD \text{ from } (13) = \frac{a \sin\theta}{2n \tan\alpha - \tan\theta};$$

therefore $CP = b\cos\alpha - \dfrac{b \sin\alpha}{2n\tan\alpha - \tan\theta}$ and $LB = b\,\dfrac{\sin(\theta-\alpha)}{\sin\theta}$.

Therefore, the length of the ray

$$=2nb\cos\alpha-a\cos\theta-\frac{b\sin\alpha}{2n\tan\alpha-\tan\theta}=2nb\cos\alpha-\frac{2na^2\cos^2\theta}{2na\cos\theta-b\cos\alpha},$$

therefore length of the caustic

$$=2nb\cos\alpha-\frac{2na^2\cos^2\theta}{2na\cos\theta-b\cos\alpha}+C\,\ldots\ldots\ldots(23).$$

The length of the ray is a *minimum* at the *outside* cusp.

For $\frac{dL}{2nd\theta}=-\frac{a\sin\alpha\cos\theta}{\cos\alpha}+\frac{2a^2\cos\theta\sin\theta}{-b\cos\alpha+2na\cos\theta}$

$$+\frac{a^2\cos^2\theta\,(a\tan\alpha\cos\theta-2na\sin\theta)}{(-)^2},$$

which vanishes when $\cos\theta=0$, that is, at the outside cusp; in which case,

$$\frac{d^2L}{2nd\theta^2}=\frac{a\sin\alpha\sin\theta}{\cos\alpha}+\frac{2a^2\sin^2\theta}{b\cos\alpha}$$

$$=a\tan\alpha+\frac{2a^2}{b\cos\alpha},$$

which is positive, as α is always positive.

The length of the ray is a *maximum* at both the *cusps in the axis*, when the light is within.

At the first cusp α and $\theta=0$, and at the second $\alpha=0$, and $\theta=\pi$, and for both, we find

$$\frac{b\,(2na\cos\theta-b)^2}{2na^2}\,\frac{d^2L}{d\theta^2}=-(n^2-1)\,a^2-3\,(b-na\cos\theta)^2,$$

and therefore, at both cusps $\frac{d^2L}{d\theta^2}$ is negative.

Hence, in estimating the length of any part, we must proceed *from* the outside cusp and *to* the cusps in the axis.

When the light is outside, at the *first* cusp $\alpha=\pi$; and at the second $\alpha=0$; in both cases θ being π; and from the general value

$$\frac{b\,(2na+b\cos\alpha)^2}{2na^2}\,\frac{d^2L}{d\theta^2}=-\cos\alpha\,\{(n^2-1)\,a^2+3\,(b\cos\alpha-na)^2\},$$

which is negative at the first and positive at the second cusp.

The length of the ray is therefore a *minimum* at the *first* cusp and a *maximum* at the second, when the light is outside.

In estimating the length of the Caustic, we must therefore proceed in this case from the first to the second cusp.

Thus, before the Caustic has an asymptote, (fig. 3.) we have

$$\text{at } A \begin{cases} \theta = \pi \\ \alpha = 0 \end{cases} \quad \text{at } B \begin{cases} \theta = \tfrac{1}{2}\pi \\ \sin\alpha = \dfrac{a}{b} \end{cases} \quad \text{at } C \begin{cases} \theta = 0; \\ \alpha = 0 \end{cases}$$

therefore length of ray at $A = 2nb + \dfrac{2na^2}{2na+b}$;

.... $B = 2n\sqrt{(b^2 - a^2)}$;

.... $C = 2nb - \dfrac{2na^2}{2na-b}$;

therefore branch $AB = 2nb + \dfrac{2na^2}{2na+b} - 2n\sqrt{(b^2-a^2)}$;

.... $BC = 2nb - \dfrac{2na^2}{2na-b} - 2n\sqrt{(b^2-a^2)}$.

When there are asymptotes, it will be sufficient to consider the length of the parts within the reflector, (fig. 4); and here it is well to observe that, as the Caustic cuts the reflector when

$$\tan\theta = (2n-1)\tan\alpha,$$

has an asymptote when

$$\tan\theta = 2n\tan\alpha,$$

and again cuts the reflector when

$$\tan\theta = (2n+1)\tan\alpha,$$

and there being a cusp when $\theta = \tfrac{1}{2}\pi$; the whole of the first asymptotic branch and the other up to the outer cusp are produced by rays to the right.

From the expression for the length of the ray,

$$L \text{ at } A = 2nb - \frac{2na^2}{2na-b};$$

$$L \text{ at } B = 2nb + \frac{2na^2}{2na+b};$$

$$L \text{ at } C \text{ the third cusp} = 2n\sqrt{(b^2-a^2)}.$$

At the intersections

$$\tan\theta = (2n \pm 1)\tan\alpha;$$

therefore, $$L = 2nb\cos\alpha - \frac{2na\cos\theta\tan\alpha}{\mp\tan\alpha}$$

$$= 2nb\cos\alpha \pm 2na\cos\theta.$$

But from (15)

$$\cos^2\theta = \frac{b^2 - a^2}{4n(n\pm 1)a^2} \text{ and } \cos^2\alpha = \frac{\overline{2n\pm 1}|^2(b^2-a^2)}{4n(n\pm 1)b^2};$$

therefore, L at intersections

$$D \text{ and } E = 2n\frac{(2n\pm 1)\sqrt{(b^2-a^2)}}{\sqrt{\{4n(n\pm 1)\}}} \pm \frac{2n\sqrt{(b^2-a^2)}}{\sqrt{\{4n(n\pm 1)\}}}$$

$$= 2\sqrt{\{n(n\pm 1)\}}\sqrt{(b^2-a^2)},$$

that is,

$$L \text{ at } E = 2\sqrt{\{n(n-1)\}}\sqrt{(b^2-a^2)},$$

$$\ldots\ldots D = 2\sqrt{\{n(n+1)\}}\sqrt{(b^2-a^2)}.$$

Hence for the parts of the caustic within the reflector,

$$AE = -2\sqrt{\{n(n-1)\}}\sqrt{(b^2-a^2)} + 2nb - \frac{2na^2}{2na-b},$$

$$B \text{ round to } C = 2nb + \frac{2na^2}{2na+b} - 2n\sqrt{(b^2-a^2)},$$

$$\text{and } CD = 2\sqrt{\{n(n+1)\}}\sqrt{(b^2-a^2)} - 2n\sqrt{(b^2-a^2)}$$

$$= 2\sqrt{(b^2-a^2)}\,[\sqrt{\{n(n+1)\}} - n].$$

Having applied the formula for the length to the two forms when the light is within, we have for the form when the light is without (fig. 6),

$$\text{length of ray at } A = \frac{2na^2}{2na-b} - 2nb,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots B = 2nb + \frac{2na^2}{2na+b};$$

therefore length of semicaustic $= 4nb - \dfrac{4na^2b}{4n^2a^2-b^2}$; which, when the light is infinitely distant $= \dfrac{4n^2-1}{n}\,b$.

If C be the point where the curve and reflector touch; the part from

$$A \text{ to } C = \sqrt{(a^2-b^2)} + 2nb - \frac{2na^2}{2na-b};$$

$$C \text{ to } B = -\sqrt{(a^2-b^2)} + 2nb + \frac{2na^2}{2na+b}.$$

The first part is imaginary, and they approach to an equality as a increases.

The form of the Caustic, at the Cusps in the Axis.

At the first, $\left.\begin{matrix}\alpha=0\\ \theta=0\end{matrix}\right\}$, at the second, $\left.\begin{matrix}\alpha=0\\ \theta=\pi\end{matrix}\right\}$;

therefore, $A=0$, $C=0$, $\sin P=0$, and $\cos P=\pm 1$;

the upper sign belonging to the first cusp.

Referring to the table of values at the beginning; we have, at the cusps,

$$\left.\begin{matrix}x=\pm B\\ y=\quad 0\end{matrix}\right\},$$

$$\left.\begin{matrix}\dfrac{dx}{dP}=0\\ \dfrac{dy}{dP}=0\end{matrix}\right\},$$

$$\left.\begin{matrix}\dfrac{d^2x}{dP^2}=\pm D\\ \dfrac{d^2y}{dP^2}=\quad 0\end{matrix}\right\}.$$

From (9), $\dfrac{d^3y}{dP^3}=-D\cos P+C\sin P-D\cos P=\mp 2D.$

Hence, if p be the increment of P from either cusp, to the point (X, Y),

$$\left.\begin{matrix}X=x\pm D\,\frac{1}{2}p^2\\ Y=\mp 2D\,\dfrac{p^3}{2.3}\end{matrix}\right\},$$

or, transferring the origin to the cusps,

$$\frac{X^3}{Y^2}=\pm\frac{9}{8}D.$$

To find D, it is only necessary to differentiate $\sin 2\theta$ in the expression for the value of C; let $C=K\sin 2\theta$;

therefore $\qquad D=\dfrac{dC}{dP}=2K\cos 2\theta\,\dfrac{d\theta}{dP}=2K\,\dfrac{d\theta}{dP},$

and $\qquad dP=2n\,d\alpha-d\theta=2n\,\dfrac{a\cos\theta}{b\cos\alpha}\,d\theta-d\theta;$

therefore $$\frac{d\theta}{dP}=\frac{b\cos\alpha}{2na\cos\theta-b\cos\alpha}=\frac{b}{\pm 2na-b},$$

and $$K=\frac{-na^2(-1)^n}{(\pm 2na-b)^3}\{(n^2-1)a^2+3(\pm na-b)^2\};$$

or, $$\frac{X^3}{Y^2}=\mp\frac{9}{4}na^2b(-1)^n\frac{(n^2-1)a^2+3(na\mp b)^2}{(2na\mp b)^4}$$

are the equations at the first and second cusps in the axis.

For parallel rays, $$\frac{X^3}{Y^2}=\mp\frac{9b}{64}(-1)^n\frac{4n^2-1}{n^3},$$

and the forms are alike, but in opposite directions.

The form at the Outer Cusp.

If (fig. 1) CD be the direction of the cusp, transform the coordinates into others (X, Y) parallel and perpendicular to CD, and let $AOD=Q$;

therefore $$\left.\begin{aligned}X&=x\sin Q-y\cos Q\\ Y&=x\cos Q+y\sin Q\end{aligned}\right\},$$

but $$\left.\begin{aligned}x&=A\sin P+B\cos P\\ y&=A\cos P-B\sin P\end{aligned}\right\}\text{ from (7);}$$

therefore $$\left.\begin{aligned}X&=-A\cos R+B\sin R\\ Y&=A\sin R+B\cos R\end{aligned}\right\},$$

by making $R=P+Q$, and therefore $\dfrac{dR}{dP}=1$;

therefore $$\frac{dX}{dP}=-B\cos R+A\sin R+\frac{dB}{dP}\sin R+B\cos R$$
$$=\left(A+\frac{dB}{dP}\right)\sin R,$$
$$\frac{dY}{dP}=B\sin R+A\cos R+\frac{dB}{dP}\cos R-B\sin R$$
$$=\left(A+\frac{dB}{dP}\right)\cos R;$$

or, $$\left.\begin{aligned}\frac{dX}{dP}&=C\sin R\\ \frac{dY}{dP}&=C\cos R\end{aligned}\right\}.$$

At the cusp,

$$\left.\begin{array}{l}\theta=\tfrac{1}{2}\pi;\ \text{and}\ \therefore A=-(-1)^n b\sin\alpha=-(-1)^n a,\ B=0,\ C=0\\ P+Q=(2n-1)\tfrac{1}{2}\pi;\ \text{and}\ \therefore \sin R=-(-1)^n\ \text{and}\ \cos R=0\end{array}\right\};$$

therefore
$$(X)=0,\ (Y)=a,$$

$$\left(\frac{dX}{dP}\right)\text{ and }\left(\frac{dY}{dP}\right)=0,$$

$$\left.\begin{array}{l}\dfrac{d^2X}{dP^2}=D\sin R+C\cos R\\ \dfrac{d^2Y}{dP^2}=D\cos R-C\sin R\end{array}\right\};$$

therefore
$$\left(\frac{d^2X}{dP^2}\right)=-(-1)^n D,$$

$$\left(\frac{d^2Y}{dP^2}\right)=0,$$

$$\frac{d^3Y}{dP^3}=\frac{dD}{dP}\cos R-D\sin R-D\sin R-C\cos R;$$

therefore
$$\left(\frac{d^3Y}{dP^3}\right)=-2D\sin R=2D(-1)^n,$$

and
$$X=-(-1)^n D\frac{p^2}{2},$$

$$Y=a+2(-1)^n D\frac{p^3}{2.3};$$

or, measuring from the cusp,

$$\left.\begin{array}{l}X=-(-1)^n D\dfrac{p^2}{2}\\ Y=(-1)^n D\dfrac{p^3}{3}\end{array}\right\};$$

therefore
$$\frac{X^3}{Y^2}=-(-1)^n D\frac{9}{8};$$

and, as before, $D=2K\cos 2\theta\dfrac{d\theta}{dP}=-2K\dfrac{d\theta}{dP}$,

and
$$K=\frac{C}{\sin 2\theta}=-\frac{3na(-1)^n}{b\cos\alpha}\text{ and }\frac{d\theta}{dP}=-1,$$

whence,
$$\frac{X^3}{Y^2}=\frac{27}{4}\,\frac{na^2}{b\cos\alpha}=\frac{27}{4}\,\frac{na^2}{\sqrt{(b^2-a^2)}};$$

The forms at all the three cusps are therefore *semicubical parabolas.*

The outer cusp being in the middle of the n^{th} reflected ray, the distances between those of successive caustics are all equal; and those produced from each half of the reflector are pointed in the same circular direction. In fig. 3, where the reflector is of very great radius, they are nearly half a circumference apart.

The Area of the n[th] *Caustic.*

The area $= b\sin\alpha\ \frac{1}{2}dL$ where $L =$ length of caustic, for $b\sin\alpha$ is the perpendicular on the tangent, and

$$L = 2nb\cos\alpha - \frac{2na^2\cos^2\theta}{2na\cos\theta - b\cos\alpha};$$

therefore

$$2\text{ area} = \int a\sin\theta dL,$$

$$= a\sin\theta L - a\int L\cos\theta d\theta,$$

$$= a\sin\theta L - 2nab\int\cos\alpha\cos\theta d\theta + 2na^2\int\frac{\cos^2\theta d\theta}{2na\cos\theta - b\cos\alpha}.$$

Now $\dfrac{\cos^2\theta}{2na\cos\theta - b\cos\alpha}$

$$= \frac{\cos^2\theta\,(2na\cos\theta + b\cos\alpha)}{4n^2a^2\cos^2\theta - b^2\cos^2\alpha} = \frac{\cos^2\theta\,(2na\cos\theta + b\cos\alpha)}{(4n^2-1)\,a^2\,(c^2 - \sin^2\theta)}$$

$$\left\{\text{where } c^2 = \frac{4n^2a^2 - b^2}{(4n^2-1)\,a^2}\right\},$$

$$= \frac{2na}{(4n^2-1)\,a^2}\left\{-\sin^2\theta + (2-c^2) + (c^2-1)\frac{1}{c^2-\sin^2\theta}\right\}$$

$$+ \frac{b\cos\alpha\cos\theta}{(4n^2-1)\,a^2}\left\{1 - (c^2-1)\frac{1}{c^2-\sin^2\theta}\right\};$$

therefore $2\text{ area} = a\sin\theta L - 2nab\int\cos\alpha\cos\theta d\theta$

$$+ \frac{4n^2a^2}{4n^2-1}\int\left\{-\sin^2\theta + (2-c^2) + \frac{(c^2-1)^2}{c^2-\sin^2\theta}\right\}d\theta$$

$$+ \frac{2nab}{4n^2-1}\int\left(1 - \frac{c^2-1}{c^2-\sin^2\theta}\right)\cos\alpha\cos\theta d\theta.$$

The several integrals which may be necessary, are

$$\int\sin^2\theta\, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

$$\int \cos\theta \cos\alpha \, d\theta = \int d\sin\theta \sqrt{\left(1 - \frac{a^2}{b^2}\sin^2\theta\right)} = \frac{b\alpha}{2a} + \frac{\sin\theta\cos\alpha}{2}$$

$$\int \frac{d\theta}{c^2 - \sin^2\theta} = \frac{1}{c\sqrt{(c^2-1)}} \tan^{-1}\left\{\frac{\sqrt{(c^2-1)}}{c}\tan\theta\right\},$$

or $$\frac{1}{c\sqrt{(1-c^2)}} \log \frac{c\cos\theta + \sqrt{(1-c^2)}\sin\theta}{\sqrt{(c^2-\sin^2\theta)}}$$

$$\int \frac{d\theta}{c^2 + \sin^2\theta} = \frac{1}{c\sqrt{(c^2+1)}} \tan^{-1}\left\{\frac{\sqrt{(c^2+1)}}{c}\tan\theta\right\}$$

$$\int \frac{\cos\theta\cos\alpha\, d\theta}{c^2 - \sin^2\theta} = \frac{a\alpha}{b} - \frac{\sqrt{(a^2c^2 - b^2)}}{bc} \tan^{-1}\left\{\frac{\sqrt{(a^2c^2-b^2)}\tan\alpha}{ac}\right\},$$

or $$\frac{a\alpha}{b} + \frac{\sqrt{(b^2 - a^2c^2)}}{bc} \log\left\{\frac{c^2\cos\alpha + c\sin\theta\sqrt{\left(1 - \frac{a^2c^2}{b^2}\right)}}{\sqrt{(c^2 - \sin^2\theta)}}\right\},$$

$$\int \frac{\cos\theta\cos\alpha\, d\theta}{c^2 + \sin^2\theta} = \frac{\sqrt{(b^2 + a^2c^2)}}{bc} \tan^{-1}\left\{\frac{\sqrt{(b^2+a^2c^2)}\tan\alpha}{ac}\right\} - \frac{a\alpha}{b}.$$

Case (1). When the luminous point is outside, or $a > b$, and therefore $c > 1$.

The limits of integration are from $\left.\begin{matrix}\theta = 0\\ \alpha = \pi\end{matrix}\right\}$ to $\left\{\begin{matrix}\theta = 0\\ \alpha = 0\end{matrix}\right.$

therefore $\int \cos\alpha\cos\theta\, d\theta = -\frac{\pi b}{2a}$, $\int \sin^2\theta\, d\theta = 0$, $\int \frac{d\theta}{c^2 - \sin^2\theta} = 0$,

and $$\int \frac{\cos\theta\cos\alpha\, d\theta}{c^2 - \sin^2\theta} = -\frac{\pi a}{b} + \frac{\pi\sqrt{(a^2c^2-b^2)}}{bc};$$

therefore 2 area

$$= \pi n b^2 + \frac{2nab}{4n^2-1}\left\{-\frac{\pi b}{2a} + \frac{\pi a}{b}(c^2 - 1) - \frac{\pi(c^2-1)\sqrt{(a^2c^2-b^2)}}{bc}\right\},$$

$$\left.\begin{aligned} c^2 - 1 &= \frac{a^2 - b^2}{(4n^2-1)a^2} \\ a^2c^2 - b^2 &= \frac{4n^2(a^2-b^2)}{4n^2-1}\end{aligned}\right\}$$

therefore 2 area

$$= \frac{2\pi n b^2(2n^2-1)}{4n^2-1} + \frac{2\pi n(a^2-b^2)}{(4n^2-1)^2} - \frac{4\pi n^2(a^2-b^2)^{\frac{3}{2}}}{(4n^2-1)^2\sqrt{(4n^2a^2-b^2)}},$$

= whole area of n^{th} caustic.

Cor. If a be infinite, then

$$\frac{(a^2-b^2)^{\frac{3}{2}}}{\sqrt{(4n^2a^2-b^2)}}=\frac{a^2}{2n}\left(1+\frac{b^2}{8n^2a^2}-\frac{3b^2}{2a^2}\right),$$

whence $$2A=\frac{4n^2-1}{4n}\pi b^2$$

is the area when the rays are parallel.

When the light is a point in the reflector,

$$\theta=\pi-\alpha, \text{ and } \cos\theta=-\cos\alpha;$$

therefore $$L=\frac{4n(n+1)}{2n+1}\cos\alpha$$

and whole area or

$$2A=\int b\sin\alpha\, dL=b\sin\alpha\, L-\frac{4n(n+1)b^2}{2n+1}\int\cos^2\alpha\, d\alpha,$$

$$=-\frac{4n(n+1)b^2}{2n+1}\left(\frac{\alpha}{2}+\frac{\sin 2\alpha}{4}\right) \text{ from } \alpha=\frac{\pi}{2} \text{ to } \alpha=0,$$

$$=\frac{n(n+1)\pi b^2}{2n+1}.$$

The caustic changes by a *flash* as the luminous point comes on the reflector from without, and changes by *another flash* as it leaves the reflector towards the centre; for in the former case, it loses one of its cusps suddenly; and in the latter, the asymptotic branches spring into existence.

Case (2). If the luminous point be within, and $2na<b$; that is, when the light is near the centre before the curve has broke into asymptotic branches,

then $\dfrac{b^2-4n^2a^2}{(4n^2-1)a^2}$ is positive; let it $=c_1^2$, and putting $-c_1^2$ for c^2 in the differential expression for the area, and observing that the limits of integration are from $\left.\begin{matrix}a=0\\ \theta=0\end{matrix}\right\}$ to $\left\{\begin{matrix}a=0\\ \theta=\pi\end{matrix}\right.$, we easily find that the whole area of the caustic

$$=\frac{4n^2a^2\pi}{4n^2-1}\left\{-\frac{1}{2}+(2+c_1^2)-\frac{(c_1^2+1)^{\frac{3}{2}}}{c_1}\right\},$$

$$=\frac{4n^2a^2\pi}{4n^2-1}\left\{\frac{3}{2}+\frac{b^2-4n^2a^2}{(4n^2-1)a^2}-\frac{(b^2-a^2)^{\frac{3}{2}}}{(4n^2-1)a^2\sqrt{(b^2-4n^2a^2)}}\right\},$$

Example to Case (1). Let $a=2b$ and $n=2$ as in fig. 6.

The area $= \pi b^2 \left\{\frac{48}{25} - \frac{16\sqrt{(21)}}{1505}\right\} = 1{\cdot}899$, the reflector being the unit.

A rough measurement gave 1.915, by clipping and weighing the fragments.

The several caustics *before* they have asymptotes are so nearly similar as to be undistinguishable; as for instance the second when $a = \frac{1}{12}$, and the twelfth when $a = \frac{1}{150}$. After losing the asymptotes, the resemblance is great, but more like that between butterflies with two, and $2n$ wings.

The Equation.

By eliminating θ from the equation to the tangent, we find

$$(x \sin 2n\alpha + y \cos 2n\alpha) \sqrt{(a^2 - b^2 \sin^2\alpha)}$$
$$+ (y \sin 2n\alpha - x \cos 2n\alpha)\, b \sin\alpha = -(-1)^n ab \sin\alpha,$$

and, by adding together the squares of (13),

$$(x^2 + y^2 - b^2 \sin^2\alpha).\{2n\sqrt{(a^2 - b^2 \sin^2\alpha)} - b \cos\alpha\}^2$$
$$= b^2 \cos^2\alpha\,(a^2 - b^2 \sin^2\alpha).$$

Also $$2 \cos 2n\alpha = (2 \cos^2\alpha)^n - 2n\,(2 \cos^2\alpha)^{n-1} + \ldots,$$
$$\sin 2n\alpha = \cos\alpha\,(2n \sin\alpha + \ldots \text{ to } \sin^{2n-1}\alpha),$$

and, by removing the radical and finding the value of $\sin 2n\alpha$ and squaring it, in order to remove $\cos\alpha$; the first equation will, if $\sin^2\alpha = z$, be of the form

$$Az^{4n+2} + Bz^{4n+1} + \ldots = 0,$$
$$Cz^4 \;\; + Dz^3 \;\; + \ldots = 0,$$

being that of the second; and, as z can be eliminated, the resulting equation will contain only $A, B, C, D\ldots$ and these quantities being rational functions of x and y, the caustics are all algebraical.

NOTE. *The width of the visible part of any caustic formed by the ring referred to, its diameter being the unit, is equal to* $\sin\frac{\pi}{2n}$ *from* (20)*; so that the nature of any visible caustic is known by inspection.*

Any tangent parallel to x, to any of these caustics, except the first, is also a tangent to all its multiple caustics (18).

ON THE INTEGRATING FACTOR OF $Pdx + Qdy + Rdz$.

By Professor De Morgan.

In the following paper, such a symbol as A_x means the partial differential coefficient of A with respect to x as therein explicitly contained.

The formula $Pdx + Qdy + Rdz$ not being integrable *per se*, and ε^M being the factor which renders it integrable, we know that M must satisfy each of the equations

$$\left.\begin{aligned} QM_x - PM_y + Q_x - P_y = 0 \\ RM_y - QM_z + R_y - Q_z = 0 \\ PM_z - RM_x + P_z - R_x = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (1),$$

the coexistence of which, that is, of the *third* simultaneously with the *other two*, is necessary or impossible, according as the following equation is or is not identically true;

$$P(R_y - Q_z) + Q(P_z - R_x) + R(Q_x - P_y) = 0 \ldots\ldots (2).$$

This well known equation was, I think, tacitly admitted to be sufficient as well as necessary, in all or most writings on the differential calculus. Cauchy (Moigno II. 492) has corrected this error of assumption. By actually going through the forms of integration of $Pdx + Qdy + Rdz = 0$, he finds that nothing stands in the way of reducing this to an equation of two variables, if equation (2) be identically true. But this is not explicitly the whole of what we want. The following brief statement relative to the criterion of two differential equations having a common solution will, I think, supply the defect.

As to equations of two variables there is little to say. When have $y' = \phi(x, y)$ and $y' = \psi(x, y)$ a common particular primitive? When $\phi_x + \phi_y \phi = \psi_x + \psi_y \psi$ is either an identical equation, or possesses a solution in common with $\phi = \psi$. I might say the latter only, because it includes the former: but we are not much accustomed, when we think of an identical equation, to remember that it has common solutions with every equation of condition. When have $y'' = \phi(x, y, y')$ and $y'' = \psi(x, y, y')$ a common particular primitive of the first order? When

$$\phi_x + \phi_y y' + \phi_{y'} \phi = \psi_x + \psi_y y' + \psi_{y'} \psi$$

is either identical, or has a common *algebraical* solution with $\phi = \psi$. And so on.

Let $AM_x + BM_y + CM_z + D = 0$ and $A'M_x + \&c. = 0$ be two partial differential equations, having A, A' &c., and M each a function of x, y, and z, and a common value of M. Since the solution M satisfies these identically, we have, by partial differentiation, six equations commencing with the terms AM_{xx}, AM_{xy}, AM_{xz}, $A'M_{xx}$, $A'M_{xy}$, $A'M_{xz}$. If we multiply these severally by $-A'$, $-B'$, $-C'$, A, B, C, and add, all the terms of the second order of differentiation disappear, and we have a *third* equation (linear) between M_x, M_y, M_z. Determine these last from the three equations, say they give $M_x = U$, $M_y = V$, $M_z = W$: then $U_y = V_x$, $V_z = W_y$, $W_x = U_z$ are the necessary and sufficient conditions of a common solution: and the common solution is $\int(Udx + Vdy + Wdz)$. But when U, V, W take the form $\frac{0}{0}$, it indicates an arbitrary function in the common solution.

To take the case with which this paper is especially concerned, let the two equations be

$$RM_y - QM_z + A = 0, \quad PM_z - RM_x + B = 0 \;\ldots.\; (3).$$

Take the results of differentiation commencing with RM_{yz}, PM_{zz}, RM_{xy}, PM_{yz}: multiply by P, Q, $-R$, $-R$, and add, which gives

$$(RR_y - QR_z)\,M_x - (RR_x - PR_z)M_y + (QP_z - PQ_z - RP_y + RQ_x)M_z$$
$$+ PA_z + QB_z - R(A_x + B_y) = 0 \ldots\ldots\ldots (4).$$

From (3) and (4) determine the values of M_x, M_y, M_z, and we have as follows. Take C from $PA + QB + RC = 0$, and let Λ be the expression in (2): then

$$\left.\begin{aligned} \Lambda M_x &= P(A_x + B_y + C_z) + B(Q_z - P_y) - C(P_z - R_x) \\ \Lambda M_y &= Q(A_x + B_y + C_z) + C(R_y - Q_z) - A(Q_x - P_y) \\ \Lambda M_z &= R(A_x + B_y + C_z) + A(P_z - R_x) - B(R_y - Q_z) \end{aligned}\right\} \ldots (5),$$

whence, in general, the common solution, if it exist, can be obtained: and by symmetry we see that if equations (3) have a common solution, then $QM_x - PM_y + C = 0$ has it also.

When $A = R_y - Q_z$, $B = P_z - R_x$, $C = Q_x - P_y$, the criteria are satisfied, but M cannot be found, since the equations (5) are identical. The single equation (2) is then a sufficient criterion of the existence of an integrating factor. We might have predicted this failure of (5) before hand: for we know that the complete integrating factor contains an arbitrary function.

The direct mode of finding the integrating factor is as follows: and it illustrates a point which all investigations connected with integrating factors seem to end in, but not directly and explicitly; namely, that the determination of an integrating factor is, not the *preliminary* of an integration, but the *integration itself*.

The condition (2) being satisfied, the equations (1) give

$$(R_y - Q_z)\,M_x + (P_z - R_x)\,M_y + (Q_x - P_y)\,M_z = 0 \ldots.(6).$$

Among the solutions of this we are to look for the factor. That is, let $a = V(x, y, z)$, $b = W(x, y, z)$ be the primitives of

$$dx : dy : dz :: R_y - Q_z : P_z - R_x : Q_x - P_y \ldots\ldots(7);$$

then the solution of (6) is $M = f(V, W)$, f being any function whatever. What forms of f will satisfy two of the equations (1)? Substitute for M in the first of these equations, giving

$$Q\,(f_V V_z + f_W W_z) - P(f_V V_y + f_W W_y) + Q_z - P_y = 0.$$

Substitute also in the second of (1). Between the two equations thus obtained, and $V = V(x, y, z)$, $W = W(x, y, z)$ eliminate x, y, z; there results an equation between V, W, f_V, f_W, the solution of which gives $f(V, W)$ in terms of V and W. This solution contains an arbitrary function of *the integral itself*. For if $\varepsilon^M Pdx + \&c.$ be dF, then $\varepsilon^M \psi F$ is the most general integrating factor of $Pdx + \&c.$, ψ being any function whatever. Should we only be able to find a particular solution of the equation between V, W, f_V, f_W, we must use it as a factor in subsequent integration, as at first contemplated. But if we obtain the complete solution, we see the integral at once, in some form of ψF.

For example take

$$yz\,(x+1)\,dx + zx\,(y+1)\,dy + xy\,(z+1)\,dz.$$

The criterion (2) is satisfied. The equation (6) is

$$x\,(z-y)\,M_x + y\,(x-z)\,M_y + z\,(y-x)\,M_z = 0.$$

The equations (7) give $V = x + y + z$, $W = xyz$. Substitution in the first of (1), of $M = f(V, W)$, gives, as it happens, the elimination of x, y, z, and produces $f_V - Wf_W = 1$. The integral of this is

$$f = V + \psi\,(V + \log W),$$

and the integral itself is $\varepsilon^{V+\log W}$ or $\varepsilon^{x+y+z}.xyz$.

It is not worth while to detail the method for any number of variables: but the following sketch of its leading points may have some interest:

A	*B*	*C*	*D*	*E*	*F*
3	9	3	1	0	2
4	16	6	4	1	3
5	25	10	10	4	4
6	36	15	20	10	5
7	49	21	35	20	6
&c.	&c.	&c.	&c.	&c.	&c.

Column *A* shows the number of independent variables in $Pdx+$&c. Column *B* shows the number of known functions P_x, P_y, Q_x, &c. Column *C* shows the number of equations of the form in (1). Column *D* shows the *apparent* number of criteria of the form (2) which must be satisfied: while *E* shews the number of these criteria which may be disregarded, as deducible from the rest; and *F*, being $C-(D-E)$ is the number of the equations (1) which must be jointly satisfied by the final process, all the rest being simultaneously satisfied, if the criteria be satisfied.

If $Ada+Bdb+Cdc$ be three of the terms, whether consecutive or not, in $Pdx+$&c. and if $(AB)^{-1}(A_b-B_a)$ be abbreviated into (ab), the criterion of the form (2) resulting from the entrance of a, b, c, is $(ab)+(bc)+(ca)=0$, which may be written $(abc)=0$. By the help of the fundamental equation $(ab)+(ba)=0$, it may easily be proved that, taking any four variables a, b, c, e, any one of the four criteria $(abc)=0$, $(bce)=0$, $(cea)=0$, $(eab)=0$, is implicitly contained in the other three. Hence, there being as many criteria immediately presented as there are triads of variables, the determination of column *E* is contained in the solution of the following question:—Given n letters, required the greatest number of triads which can be taken, under the condition that one triad out of every quaternion shall be missing. The answer is $\frac{1}{2}(n-1)(n-2)$, which is $D-E$. This is easily proved by successive induction.

THEOREMS ON POLAR CONICS WITH RESPECT TO CURVES OF THE THIRD CLASS.

By the Rev. T. St. Lawrence Smith, B.A.

A curve of the third class may be of the third, fourth, or sixth degree; if of the third degree it has one point of inflexion and one cusp, if of the fourth, one double tangent and three cusps, if of the sixth, nine cusps—of which six must be imaginary.

Every line has, with respect to such a curve given by its tangential equation, a pole and a polar conic; the ordinary theorems about polar conics of points with respect to curves of the third degree give easily, when their equations are interpreted as tangential ones, an equal number of theorems about the polar conics of lines with respect to curves of the third class, of which theorems the following are a few. (See Salmon's *Higher Plane Curves*, p. 284.)

The envelope of all lines whose poles lie on a given line is the polar conic of that line; therefore

All lines having their poles at infinity touch the polar conic of the line at infinity, i.e. *the central conic*, or since the first theorem may be thus stated:

If a polar conic touch a fixed line, its polar line passes through the pole of that fixed line.

We have as a particular case,

The polar lines of all polar parabolas intersect in the centre of the curve.

Since the pole of a line with respect to the curve must be likewise its pole with respect to the polar conic of the line, it follows that

The centre of the curve is also the centre of the central conic.

If the central conic be a hyperbola, its asymptotes are therefore lines whose poles are at infinity, and whose polar conics are parabolas, hence they must each be diameters of their own polar parabolas.

If the curve have a double tangent every polar conic touches it; therefore,

If the curve have a double tangent at infinity, every polar conic is a parabola.

In this case the centre becomes indeterminate, and the central conic breaks up into the two points in which the line at infinity touches the curve.

If the curve have a point of inflexion, every polar conic touches the tangent at the point of inflexion.

Hence,

If the curve have a point of inflexion at infinity, no line with respect to it can have a polar ellipse or parabola, unless the tangent be also at infinity, in which case every polar conic will be a parabola whose diameters will be parallel to a fixed line.

The locus of the poles of all lines passing through a point is a conic, which may be called the polar conic of that point, it is likewise the envelope of the polar lines of all the polar conics passing through the point.

The polar conic of a point is also the envelope of the polar lines of that point with regard to the polar conics of all lines passing through the point.

One of the most interesting cases (and the simplest case) of curves of the third class, is when they are of the third degree likewise, that is when they have a point of inflexion and a cusp.

Such a curve may be represented by the *same* equation in both tangential and trilinear coordinates,

$$\alpha^2\beta - k\gamma^3 = 0.*$$

In tangential coordinates,

$\beta = 0$, the point B, is the cusp,

$\beta = 0$, $\gamma = 0$, the line BC, its tangent,

$\alpha = 0$, the point A, the point of inflexion,

$\alpha = 0$, $\gamma = 0$, the line AC, its tangent.

In trilinear coordinates, (fig. 7)

$\alpha = 0$, $\gamma = 0$, the point B, is the cusp,

$\alpha = 0$, the line BC, its tangent,

$\beta = 0$, $\gamma = 0$, the point A, the point of inflexion,

$\beta = 0$, the line AC, its tangent.

We shall call for distinctness, the polar conic of any line or point, with reference to tangential coordinates, the tan-

* When the same curve is expressed in both trilinear and tangential coordinates by the same equation

$$\alpha^2\beta - k\gamma^3 = 0,$$

k will almost always be a *different* quantity in each system.

gential polar conic; with reference to trilinear coordinates, the trilinear polar conic: polar lines and points, with respect to the two systems, we shall also distinguish similarly.

The equation of the tangential pole of any line $(\alpha', \beta', \gamma')$ is

$$2\alpha'\beta'\alpha + \alpha'^2\beta - 3k\gamma'^2\gamma = 0.$$

If the line be at infinity its equations are

$$\alpha = \beta = \gamma,$$

and therefore its pole or the centre of the curve is given by the equation

$$2\alpha + \beta - 3k\gamma = 0.$$

This point is easily found,

Cut BA in E so that $BE = 2EA$, join the points C and E, and cut the joining line in D, so that $ED : DC = -k$; the point D is the centre of the curve.

If k be positive and less than unity, D lies on the side of AB remote from C; if positive and greater than unity, C is between E and D; if negative, D lies between E and C; if $k = 1$, D is at infinity.

The two trilinear centres of the curve (which having a cusp has but two exclusive of the cusp itself) lie on the same line EC, and with the points E and C cut that line harmonically.

The equation

$$2\alpha'\beta'\alpha + \alpha'^2\beta - 3k\gamma'^2\gamma = 0$$

now represents the trilinear polar of the point $(\alpha', \beta', \gamma')$, the the trilinear poles of the line

$$l\alpha + m\beta + n\gamma = 0$$

are consequently given by the equations

$$\frac{2\alpha\beta}{l} = \frac{\alpha^2}{m} = -\frac{3k\gamma^2}{n};$$

if the line be that at infinity,

$$l : m : n = \sin A : \sin B : \sin C,$$

and therefore

$$\frac{2\alpha\beta}{\sin A} = \frac{\alpha^2}{\sin B} = -\frac{3k\gamma^2}{\sin C},$$

equivalent to

$$2\beta \sin B = \alpha \sin A,$$

the line CE, and

$$\frac{\alpha^2}{\sin B} + 3k\frac{\gamma^2}{\sin C} = 0$$

two lines, real or imaginary, of the form

$$\alpha = \pm p\beta,$$

and therefore forming with the lines $\alpha = 0$, $\beta = 0$ a harmonic pencil.

It can be easily seen that the other tangential polar of the centre D is the line joining the middle points of the sides BC and AC of the triangle BCA.

The following table, which can be easily understood, gives one or two of the most obvious properties of the polar conics of a point and a line in both systems, and shows that the tangential and trilinear polar conic of the same point, or the same line, can never coincide.

TRILINEAR COORDINATES, POINT.	TANGENTIAL COORDINATES, LINE.
$(\alpha', \beta', \gamma')$	
POLAR CONIC. $\alpha(2\alpha'\beta + \beta'\alpha) - 3k\gamma'\gamma^2 = 0.$	
Touches BC (tangent to cusp) at B the cusp.	Touches AC (stationary tangent) at A (point of inflexion).
LINE.	POINT.
$l\alpha + m\beta + n\gamma = 0.$	
POLAR CONIC. $3km\gamma(m\beta - 2l\alpha) + n^2\alpha^2 = 0.$	
Passes through C (intersection of stationary tangent with tangent to cusp).	Touches AB (line joining cusp and point of inflexion).
Touches line AB (line joining cusp and point of inflexion) at B (cusp).	Touches AC (stationary tangent) at C (intersection of stationary tangent and tangent to cusp).

Precisely as in the case of trilinear polar conics, it can be shown that the tangential polar conic of a *point*, with respect to *any* curve of the third class, will be a hyperbola, when *real* tangents can be drawn from it to the central conic, and that when the point lies on the central conic itself, its tangential polar conic will be a parabola.

The condition that the tangential polar conic of a *line* should be a hyperbola is, that when the coordinates of the line have been substituted in

$$A=\frac{d^2U}{dx^2},\quad B=\frac{d^2U}{dx\,dy},\quad C=\frac{d^2U}{dy^2dz},$$

$$D=\frac{d^2U}{dz\,dx},\quad E=\frac{d^2U}{dy\,dz},\quad F=\frac{d^2U}{dz^2},$$

the functions

$$A+2B+C+2D+2E+F,$$

and

$$AE^2+CD^2+FB^2-ACF-2BDE,$$

should have the *same* sign; if they have different signs the conic will be an ellipse, if

$$AE^2+CD^2+FB^2-ACF-2BDE=0,$$

it will break up into two points, if

$$A+2B+C+2D+2E+F=0,$$

(*i.e.* if the line pass through the centre) it will be a parabola. (See Salmon's *Higher Plane Curves*, p. 8.)

I have not been able to find a geometrical interpretation of this condition in the general case of any curve of the third class, but for the curve I have been hitherto principally considering, of the third class and third degree, such an interpretation is easy.

Taking, as before, the points A, B, C,

$$\alpha=0,\quad \beta=0,\quad \gamma=0,$$

as the points of reference, the equation of the curve being

$$U=\alpha^2\beta-k\gamma^3=0,$$

we have

$$A=\frac{d^2U}{d\alpha^2}=\ \ 2\beta,\quad E=\frac{d^2U}{d\beta\,d\gamma}=0,$$

$$C=\frac{d^2U}{d\beta^2}=\ \ 0,\quad D=\frac{d^2U}{d\gamma\,d\alpha}=0,$$

$$F=\frac{d^2U}{d\gamma^2}=-6k\gamma,\quad B=\frac{d^2U}{d\alpha\,d\beta}=2\alpha,$$

therefore

$$A+2B+C+2D+2E+F=2(\beta+2\alpha-3k\gamma),$$

$$AE^2+CD^2+FB^2-ACF-2BDE=-24k\gamma\alpha^2,$$

the condition that the conic should be a hyperbola, therefore is that

$$2\alpha+\beta-3k\gamma \text{ and } k\gamma$$

should have *different* signs.

Now $\dfrac{2\alpha+\beta-3k\gamma}{3(1-k)}$ is the perpendicular on the polar line from the centre, and γ the perpendicular on the same line from C.

The different cases must be distinguished according as k is negative, positive and less than unity, or positive and greater than unity.*

$k<0$. The centre D lies between C and E.

The perpendicular from the centre has the same sign as $2\alpha+\beta-3k\gamma$, but the perpendicular from C has a different sign from $k\gamma$; in order that the conic should be a hyperbola, these perpendiculars should have different signs, the line must therefore *not* cut CE between C and D.

$k>0<1$. E lies between C and D.

Both perpendiculars have the same signs as $2\alpha+\beta-3k\gamma$ and $k\gamma$ respectively, they must therefore have different signs from each other, the line must therefore cut CD between C and D.

$k>1$. C lies between D and E.

The perpendicular from D has a different sign from $2\alpha+\beta-3k\gamma$, while the perpendicular from C has the same sign as $k\gamma$, the perpendiculars must therefore have the same sign, *i.e.* the line must not cut DE between D and C.

All these cases are included in the following statement: *A point moving from* D *to* C *in such a direction as to reach* E *before* C, *passes over all that part of the right line* DC, *which every right line must cut whose tangential polar conic is a hyperbola.*

In the value of

$$AE^2+CD^2+FB^2-ACF-2BDE,$$

* I have accidently omitted the case of $k=1$, the perpendicular from the centre D being then infinite cannot be compared in sign with the perpendicular from C; but the geometric condition requisite that the conic should be a hyperbola may be easily ascertained thus, the analytic condition is, that

$$2\alpha+\beta-3\gamma \text{ and } \gamma$$

should have different signs. $2\alpha+\beta$ is three times the perpendicular from E, which call ϵ, therefore $3(\epsilon-\gamma)$ and consequently $\epsilon-\gamma$ must have a different sign from γ, which will happen when the polar line cuts CE on the same side of C on which E lies, and therefore this case also comes under the general statement.

we suppressed the factor α^2 as it is always positive for real values, and it was the sign only that we were concerned with; but, if the line pass through A, then for such line $\alpha = 0$ and the polar conic will be found to be always two points on the line AC.

If the line pass through C, its polar conic breaks up into the point A, and some other point on BA; if the line be CD itself, this other point goes off to infinity on the line BA.

If the line pass through D its polar conic is a parabola, excepting, of course, the lines CD and AD; the polar conic of the latter of these lines is the middle point of CA, and the point at infinity on the same line.

Crossmaylen, *9th March*, 1858.

ON RODRIGUES' METHOD FOR THE ATTRACTION OF ELLIPSOIDS.

By A. Cayley.

THE following is in substance the method given in the "Memoire sur l'attraction des Spheroïdes," par M. Rodrigues, *Corresp. sur l'Ecole Polyt.*, t. III. pp. 361–385 (1815). It will be seen that the method is very similar to that given two years before by Gauss, see my paper "On Gauss' Method for the Attraction of Ellipsoids," *Journal*, t. I. pp. 162–166: the solution in fact depends upon the geometrical theorem therein quoted, viz. if M be any point, P a point of a closed surface, PQ the normal (lying outside the surface) at the point P, dS the element of the surface at that point, and if MQ denotes the angle MPQ and $\overline{MP}$ the distance of the points M and P, then, theorem, the integral

$$\iint \frac{dS \cos MQ}{\overline{MP}^2}$$

has for its value

$$0, \ -2\pi \ \text{or} \ -4\pi$$

according as M is exterior to, upon, or interior to the surface.

Suppose that M is the attracted point and taking A, B, C for the semiaxes of the surface of the attracting ellipsoid, or, if we please, for any semiaxes of an arbitrary ellipsoidal surface confocal with the surface of the attracting ellipsoid, let P be a point on the surface of the interior similar ellipsoid whose semiaxes are rA, rB, rC. The coordinates of M are taken to be a, b, c, and those of P are taken to be x, y, z, and the value of the potential is

$$V=\int\frac{dm}{MP},$$

where dm is the element of mass.

We may write

$$x=rA\xi,$$
$$y=rB\eta,$$
$$z=rC\zeta,$$

and then ξ, η, ζ will be the coordinates of a point P' on the surface of a sphere radius unity corresponding in a definite manner to the point P on the surface of the internal similar ellipsoid. And if $d\sigma$ be the element of the spherical surface, then we have

$$dm=ABCr^2dr\,d\sigma,$$

and therefore

$$\frac{V}{ABC}=\int\frac{r^2dr\,d\sigma}{MP};$$

where, in order to obtain the value of the potential V for the ellipsoid whose semiaxes are A, B, C, the integrations must be extended over the spherical surface and from $r=0$ to $r=1$.

Suppose that dS is the element of the internal similar surface at P, and let p be the perpendicular from the centre upon the tangent plane at P, we have

$$dS=\frac{r^3ABC}{p}\,d\sigma.$$

Let P_0 be the point on the ellipsoid (A, B, C) similarly situated to the point P on the ellipsoid (rA, rB, rC); the coordinates of P_0 are $A\xi$, $B\eta$, $C\zeta$; and if p_0 be the perpendicular from the centre upon the tangent plane at P_0, then $p=rp_0$, and the preceding equation becomes

$$dS=\frac{r^2ABC}{p_0}\,d\sigma.$$

Imagine now an ellipsoidal surface confocal with the surface (A, B, C) and having for its semiaxes

$$A+\delta A,\quad B+\delta B,\quad C+\delta C,$$

and let P_0' be the point on this surface which *corresponds* with the point P_0 on the surface (A, B, C); that is, let P_0' be the point whose coordinates are

$$(A+\delta A)\,\xi, \quad (B+\delta B)\,\eta, \quad (C+\delta C)\,\zeta;$$

and let P' be in like manner the point whose coordinates are

$$r\,(A+\delta A)\,\xi, \quad r\,(B+\delta B)\,\eta, \quad r\,(C+\delta C)\,\zeta;$$

the points P, P' will be in like manner *corresponding* points on the surface (rA, rB, rC) and on the confocal surface $\{r\,(A+\delta A),\ r\,(B+\delta B),\ r\,(C+\delta C)\}$; and if the normal distance at the point P_0 of the first two surfaces is δN, then the normal distance at the point P of the second two surfaces will be $r\delta N$. The decrement of $\overline{MP}$ will be equal to the normal distance $r\delta N$ of the two surfaces at the point P multiplied into the cosine of the angle MQ, and we have, by a property of confocal surfaces,

$$A\delta A = B\delta B = C\delta C = p_0\delta N = \text{(suppose) } \tfrac{1}{2}\delta\theta,$$

we have therefore

$$\delta\overline{MP} = -\frac{\tfrac{1}{2}r\delta\theta}{p_0}\cos MQ.$$

Hence from the equation

$$\frac{V}{ABC} = \int \frac{r^2\,dr\,d\sigma}{\overline{MP}}.$$

We deduce

$$\delta\,\frac{V}{ABC} = \int r^2\,dr\,d\sigma\,\frac{\tfrac{1}{2}r\delta\theta}{p_0}\,\frac{\cos MQ}{\overline{MP}^2}.$$

But we have

$$\frac{r^2 d\sigma}{p_0} = \frac{dS}{ABC}.$$

And the equation thus becomes

$$\delta\,\frac{V}{ABC} = \frac{\tfrac{1}{2}\delta\theta}{ABC}\int r\,dr\,\frac{dS\cos MQ}{\overline{MP}^2}.$$

It may be proper to remark here by way of recapitulation that the course of the investigation has been as follows: viz. that with a view to obtaining the potential V of an attracting ellipsoid we have found the increment of $\dfrac{V}{ABC}$ in passing from the ellipsoidal surface (A, B, C) to the ellipsoidal surface $(A+\delta A,\ B+\delta B,\ C+\delta C)$, each of them *con-*

focal with the surface of the attracting ellipsoid; and that for finding such increment we have had to consider the two surfaces (rA, rB, rC) and $\{r(A+\delta A), r(B+\delta B), r(C+\delta C)\}$ confocal to each other and respectively *similar* to the first mentioned two surfaces.

Resuming the formula just obtained, the integral with respect to dS, is taken over the entire surface of the internal similar ellipsoid (rA, rB, rC), and if the attracted point M is external to the ellipsoid (A, B, C) it will be external to the interior similar ellipsoid (rA, rB, rC): hence in this case the double integral vanishes for all values of r, or we have

$$\delta \frac{V}{ABC} = 0;$$

that is the function $V \div ABC$ which represents the ratio of the potential to the mass, is not altered in passing from the ellipsoid (A, B, C) to the confocal ellipsoid

$$(A+\delta A,\ B+\delta B,\ C+\delta C),$$

or, what is the same thing, the potentials (and therefore the attractions) of confocal ellipsoids upon the same external point are proportional to their masses; this is in fact Maclaurin's theorem for the attraction of ellipsoids upon an external point.

But if the attracted point M is interior to the ellipsoid (A, B, C) then writing

$$\frac{a^2}{A^2} + \frac{b^2}{B^2} + \frac{c^2}{C^2} = r'^2,$$

where r' is less than unity, the double integral is 0 from $r=0$ to $r=r'$ and -4π from $r=r'$ to $r=1$, and we have

$$\delta \frac{V}{ABC} = -\frac{2\pi\delta\theta}{ABC}\int_{r'}^{1} r dr$$

$$= -\frac{\pi\delta\theta}{ABC}(1-r'^2)$$

$$= \frac{\pi\delta\theta}{ABC}\left(\frac{a^2}{A^2} + \frac{b^2}{B^2} + \frac{c^2}{C^2} - 1\right);$$

that is, the right-hand side of the equation is the increment (or taken with its sign reversed so as to be positive, it is the decrement) of the function $V \div ABC$ in passing from the ellipsoid (A, B, C) to the confocal ellipsoid

$$(A+\delta A,\ B+\delta B,\ C+\delta C),$$

where $\frac{1}{2}\delta\theta = A\delta A = B\delta B = C\delta C.$

The preceding formula gives at once the potential for an interior point, in fact taking α, β, γ for the semiaxes of the ellipsoid and writing

$$A^2 = \alpha^2 + \theta, \quad B^2 = \beta^2 + \theta, \quad C^2 = \gamma^2 + \theta,$$

and using the ordinary symbol d instead of δ, we have

$$\frac{d}{d\theta}\,\frac{V}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}$$

$$= \frac{\pi}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\},$$

and integrating from $\theta = 0$ to $\theta = \infty$, we have

$$-\frac{V}{\alpha\beta\gamma} = \pi\int_0^\infty \frac{d\theta}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\}.$$

where V is now the potential for the ellipsoid whose semiaxes are α, β, γ; and we have therefore

$$V = -\pi\alpha\beta\gamma\int_0^\infty \frac{d\theta}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\}.$$

To find the potential for an external point it is only necessary to remark that by the theorem above demonstrated $V \div \alpha\beta\gamma$ is equal to the corresponding function for the confocal ellipsoid through the attracted point, that is for the ellipsoid whose semiaxes are $\sqrt{(\alpha^2 + \theta_1)}$, $\sqrt{(\beta^2 + \theta_1)}$, $\sqrt{(\gamma^2 + \theta_1)}$, where θ_1 is a positive quantity such that

$$\frac{a^2}{\alpha^2 + \theta_1} + \frac{b^2}{\beta^2 + \theta_1} + \frac{c^2}{\gamma^2 + \theta_1} = 1,$$

that is in the value of $V \div \alpha\beta\gamma$ we have only to write the above values in the place of α, β, γ; and if we then write $\theta - \theta_1$ in the place of θ the limits will be ∞, θ_1, and the expression for the potential is

$$V = -\pi\alpha\beta\gamma\int_{\theta_1}^\infty \frac{d\theta}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\},$$

which completes the investigation.

NOTE ON THE THEORY OF ATTRACTION.

By A. Cayley.

Imagine a closed surface, the equation of which contains the two parameters m, h. Call this the surface (m, h), and suppose also that for shortness the shell of uniform density included between the surfaces (m, h), (n, h) is called the shell (m, n, h). Suppose now that the surface is such—

1°. That the infinitesimal shell $(m, m+dm, h)$ exerts no attraction upon an internal point.

2°. That the equipotential surfaces of the shell in question for external points are the surfaces (m, k), where k is arbitrary.

Then, first, the attraction of the shell on a point of the equipotential surface (m, k) is proportional to the normal thickness at that point of the shell $(m, m+\delta m, k)$ or more precisely taking the density of the attracting shell as unity the attraction is $=4\pi\times$ mass of shell $(m, m+dm, h)$ into normal thickness of shell $(m, m+\delta m, k)$ divided by mass of the last mentioned shell.

In fact the shell $(m, m+\delta m, k)$ exerts no attraction on an internal point, consequently if over the surface (m, k) we distribute the mass of the original shell $(m, m+dm, h)$ in such manner that the density at any point is proportional to the normal thickness of the shell $(m, m+\delta m, k)$ the distribution will be such that the attraction on an internal point may vanish; but in order that this may be the case, the density must be equal to $\frac{1}{4\pi}$ into the attraction upon that point of the shell $(m, m+\delta m, k)$. Hence the attraction is proportional to the normal thickness, and if the whole mass distributed over the surface (m, k) is precisely equal to the mass of the shell $(m, m+dm, h)$, then the density at any point must be equal to the mass into normal thickness divided by mass of $(m, m+\delta m, k)$, and attraction $=4\pi$ into density $=4\pi\times$ mass of shell $(m, m+dm, h)$ into normal thickness of shell $(m, m+\delta m, k)$ divided by mass of the last mentioned shell.

And, secondly, the attractions of the solids bounded by the two surfaces (n, h), (n, h_1) respectively upon the same exterior point are proportional to their masses.

For the solid (n, h) may be divided into shells $(m, m+dm, h)$ and for this shell the equipotential surface is (m, k) and the attraction of the shell varies as mass of $(m, m+dm, h)$ into normal thickness of the shell $(m, m+\delta m, k)$. But in like manner the solid (n, h_1) may be divided into shells $(m, m+dm, h_1)$ and the attraction of the shell varies as mass of $(m, m+dm, h_1)$ into normal thickness of the shell $(m, m+\delta m, k)$ and the attractions are in each case in the direction normal to the shell (m, k), and therefore in the same direction; that is, the attraction of the shell $(m, m+dm, h)$ is in the same direction as that of the shell $(m, m+dm, h_1)$ and the two attractions are proportional to the masses. Hence integrating from $m=0$ (if for this value the included space is zero) to $m=n$, the attractions of the solids (n, h), (n, h_1) are composed of elements proportional and parallel, the elements of the attraction of (n, h) to the elements of the attraction of (n, h_1); and consequently the total attractions are in the same direction and proportional to the masses. Thirdly, the attractions of the two surfaces upon the same interior point are equal.

A surface having the properties in question is of course the ellipsoidal surface

$$\frac{x^2}{m(a^2+h)}+\frac{y^2}{m(b^2+h)}+\frac{z^2}{m(c^2+h)},$$

where if m varies (h being constant) the several surfaces are similar to each other, but if h varies (m being constant) the several surfaces are confocal to each other: for it is in fact well known that the infinitesimal shell bounded by similar ellipsoidal surfaces has the properties assumed with respect to the shell $(m, m+dm, h)$. The first theorem in effect reduces the problem of the determination of an ellipsoid upon an exterior point to a single integration, and constitutes the foundation of Poisson's method for the attraction of ellipsoids. The second theorem (Maclaurin's theorem for the attraction of ellipsoids on the same external point) shews that the attraction of an ellipsoid upon an external point can be found by means of the attraction of the confocal ellipsoid through the attracted point; and by the third theorem the attraction of an ellipsoid upon an interior point is equal to that of the similar ellipsoid through the attracted point; hence the second and third theorems reduce the determination of the attraction of an ellipsoid upon an external or internal point to that of an ellipsoid upon a point on the surface.

2, Stone Buildings, W.C.,
7th April, 1858.

ON THE CLASSIFICATION OF POLYGONS OF A GIVEN NUMBER OF SIDES.

By Professor De Morgan.

Let any number be distributed into parts in the several compartments of what looks like a genealogical table. Thus (fig. 8) is one way of distributing 48.

This is one type, as we shall see, of a 50-gon. This one type may be varied in form, without effective variation of meaning, by taking any one as the common ancestor, and proceeding through all the degrees of affinity, in the following way. Say that the first 3 on the right, in what is now the second descent, shall be made the source: then from this 3 springs only 5; from 5 spring 5, 2, 1; from the last named 5 spring 2, 3, 1; from 2 spring 3, 2; from 1 springs 2; and so on.

And it is a condition that no number n is to have more than $n+1$ children, except the common ancestor, who may have $n+2$: that is, each component may have $n+2$ others in the first degree of connexion.

Every number n answers to a *simple* polygon of $n+2$ sides: that is, a polygon in which no side cuts another side. The simple polygon may be either convex, or with some angles re-entrant. The figure shews one variety of the polygon of $48+2$ sides, constructed as follows.

When two polygons have, not a common angle, but a common vertex with opposite angles, so that two contiguous sides of either are in two contiguous sides of the other produced, let either be said to be *surmounted* by the other. Every complex polygon, in which some sides cut others, can be constructed out of simple polygons, each of which is surmounted by others, so that no two connected polygons surmount each other at more than one angle. And every way of distributing n, after the preceding manner, shews one variety of the *autotomic* polygon of $n+2$ sides. Thus, consulting the preceding table, we see that one variety of the 50-gon is formed by surmounting a 7-gon at four angles by a 7-gon, 4-gon, 5-gon, and 3-gon: the 7-gon being surmounted at three angles (not yet used) by a 4-gon, a 3-gon, and a 5-gon: and so on.

To take more simple cases. A star-shaped pentagon is a re-entrant quadrilateral surmounted at the re-entrant angle by a triangle of sufficient size. A star-shaped octagon is a rhombus surmounted at opposite angles by two re-entrant quadrilaterals: it is also a hexagon with two re-entrant angles, at each of which it is surmounted by a triangle.

The varieties under each species depend upon the angles chosen, and the manner in which the sides cut one another. When it happens that one figure surmounts another at two or more angles, the polygon degenerates, losing two sides for each angle of surmountance which is added to those of the specific type. Thus the star-shaped pentagon is a triangle surmounted by two triangles, each of which is surmounted by one more triangle, these last triangles also surmounting each other. But for this last occurrence we should have had a heptagon: as it is, we have a pentagon.

I was led to what precedes some years ago in an attempt to classify what appeared the never-ending hexagons of Pascal's theorem.

ON THE MOTION OF A BODY REFERRED TO MOVING AXES.

By G. M. Slesser, B.A., Queens' College.

1. In problems relating to the equilibrium of a rigid body, the solution is in general much simplified, by a particular choice of the coordinate axes, to which the position of the body and the points of application and direction of the forces are referred. A similar simplification of problems in the motion of a body, cannot be immediately effected, because the axes which it would be most convenient to use as coordinate axes will in general themselves be moveable, and the ordinary equations of motion which refer to fixed coordinate axes, will require to be replaced by others which apply to the case of moving axes. It is the object of this paper to investigate such equations of motion for a particle, and to give some useful transformations for determining the motion of a rigid body by means of moveable axes.

2. The position of a particle P is given by its coordinates x, y, z referred to a system of rectangular axes meeting in a fixed point O, and turning about it, so as always to be parallel to a system of axes fixed in a body having component angular velocities $\omega_1, \omega_2, \omega_3$ about those axes.

With centre O describe a sphere of radius unity, and let the coordinate axes meet it in the points A, B, C, and a fixed straight line through O in the point L.

Also let $\angle LOA = \alpha, \quad LOB = \beta, \quad LOC = \gamma.$

Then the projection of OP on OL is $x\cos\alpha + y\cos\beta + z\cos\gamma$; therefore the velocity parallel to OL is

$$\frac{d}{dt}(x\cos\alpha + y\cos\beta + z\cos\gamma)$$

$$= \frac{dx}{dt}\cos\alpha + \frac{dy}{dt}\cos\beta + \frac{dz}{dt}\cos\gamma - x\sin\alpha\frac{d\alpha}{dt} - y\sin\beta\frac{d\beta}{dt} - z\sin\gamma\frac{d\gamma}{dt}.$$

And when OL coincides with OA, or $\alpha = 0$, $\beta = \gamma = \frac{1}{2}\pi$, this gives the velocity parallel to the axis of x, which is therefore

$$\frac{dx}{dt} - y\frac{d\beta}{dt} - z\frac{d\gamma}{dt}.$$

And $\frac{d\beta}{dt}$ is the velocity with which B moves from a fixed point coinciding with the instantaneous position of A; therefore

$$\frac{d\beta}{dt} = \omega_3,$$

and similarly

$$\frac{d\gamma}{dt} = -\omega_2;$$

therefore the velocity is

$$\frac{dx}{dt} - y\omega_3 + z\omega_2,$$

hence the component velocities u, v, w parallel to the axes are

$$\left.\begin{aligned} u &= \frac{dx}{dt} - y\omega_3 + z\omega_2 \\ v &= \frac{dy}{dt} - z\omega_1 + x\omega_3 \\ w &= \frac{dz}{dt} - x\omega_2 + y\omega_1 \end{aligned}\right\} \quad \text{..................} \quad (1).$$

Again the velocity parallel to the fixed line OL is

$$u\cos\alpha + v\cos\beta + w\cos\gamma;$$

therefore the acceleration parallel to OL is

$$\frac{du}{dt}\cos\alpha + \frac{dv}{dt}\cos\beta + \frac{dw}{dt}\cos\gamma - u\sin\alpha\frac{d\alpha}{dt} - v\sin\beta\frac{d\beta}{dt} - w\sin\gamma\frac{d\gamma}{dt},$$

and when $\alpha = 0$, $\beta = \gamma = \frac{1}{2}\pi$, this becomes in the same manner as before

$$\frac{du}{dt} - v\omega_3 + w\omega_2,$$

hence the accelerations parallel to the three coordinate axes are

$$\left.\begin{array}{l} \dfrac{du}{dt} - v\omega_3 + w\omega_2 \\ \dfrac{dv}{dt} - w\omega_1 + u\omega_3 \\ \dfrac{dw}{dt} - u\omega_2 + v\omega_1 \end{array}\right\} \text{.................... (2),}$$

where the values of u, v, w are given by (1).

In the particular case in which the axes of x, y move round in a plane, with an angular velocity ω, we have to put $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$, in the above formulæ, and we have for the velocities parallel to those axes

$$u = \frac{dx}{dt} - y\omega \quad \text{and} \quad v = \frac{dy}{dt} + x\omega \text{............ (3),}$$

and for the accelerations the expressions

$$\frac{du}{dt} - v\omega, \quad \frac{dv}{dt} + u\omega \text{.................. (4),}$$

formulæ already known.

In the general case, if X, Y, Z be the components of the impressed force parallel to the coordinate axes, we have the following equations for the motion of a particle mass m

$$\left.\begin{array}{l} m\left(\dfrac{du}{dt} - v\omega_3 + w\omega_2\right) = X \\ m\left(\dfrac{dv}{dt} - w\omega_1 + u\omega_3\right) = Y \\ m\left(\dfrac{dw}{dt} - u\omega_2 + v\omega_1\right) = Z \end{array}\right\} \text{..............(5),}$$

u, v, w being given by (1).

3. A body moving about a fixed point, has angular velocities Ω_1, Ω_2, Ω_3 about a system of axes fixed in space, and ω_1, ω_2, ω_3 are its angular velocities referred to a system of axes fixed in the body, to express $\dfrac{d\Omega_1}{dt}$, $\dfrac{d\Omega_2}{dt}$, $\dfrac{d\Omega_3}{dt}$ in

terms of $\omega_1, \omega_2, \omega_3$ when the axes fixed in space are chosen so as to coincide at the time t, with the axes fixed in the body.

Let as before the axes fixed in the body meet the surface of the sphere in A, B, C, and OL be a fixed direction,

$$\angle LOA = \alpha, \text{ \&c.}$$

then if $\Omega =$ angular velocity about the fixed line OL,

$$\Omega = \omega_1 \cos\alpha + \omega_2 \cos\beta + \omega_3 \cos\gamma;$$

therefore

$$\frac{d\Omega}{dt} = \frac{d\omega_1}{dt}\cos\alpha + \ldots - \omega_1 \sin\alpha \frac{d\alpha}{dt} - \omega_2 \sin\beta \frac{d\beta}{dt} - \omega_3 \sin\gamma \frac{d\gamma}{dt},$$

when $\alpha = 0$, $\beta = \gamma = \frac{1}{2}\pi$, this becomes

$$\frac{d\Omega_1}{dt} = \frac{d\omega_1}{dt} - \omega_2 \frac{d\beta}{dt} - \omega_3 \frac{d\gamma}{dt},$$

and $$\frac{d\beta}{dt} = \omega_3, \qquad \frac{d\gamma}{dt} = -\omega_2;$$

therefore $\dfrac{d\Omega_1}{dt} = \dfrac{d\omega_1}{dt}$ so $\dfrac{d\Omega_2}{dt} = \dfrac{d\omega_2}{dt}$ and $\dfrac{d\Omega_3}{dt} = \dfrac{d\omega_3}{dt}$ .... (6),

as proved in Griffin's *Rigid Dynamics*, Art. 78, by a rather long analytical process.

4. A body moving about a point has angular velocities $\Omega_1, \Omega_2, \Omega_3$ about a system of axes fixed or moveable, and $\omega_1, \omega_2, \omega_3$ are its angular velocities about another system of coordinate axes moving in a given manner relatively to the other system to express $\dfrac{d\Omega_1}{dt}$, &c. in terms of $\omega_1, \omega_2, \omega_3$ when the first system of axes is so chosen as to coincide instantaneously at the time t with the second system.

Using the same construction as before, if OL be any line fixed or moveable and Ω the angular velocity about it, we have

$$\Omega = \omega_1 \cos\alpha + \omega_2 \cos\beta + \omega_3 \cos\gamma,$$

$$\frac{d\Omega}{dt} = \frac{d\omega_1}{dt}\cos\alpha + \ldots - \omega_1 \sin\alpha \frac{d\alpha}{dt} - \text{\&c.},$$

and when $\alpha = 0$, $\beta = \gamma = \frac{1}{2}\pi$, we have

$$\frac{d\Omega}{dt} = \frac{d\omega_1}{dt} - \omega_2 \frac{d\beta}{dt} - \omega_3 \frac{d\gamma}{dt}.$$

Now $\frac{d\beta}{dt}$ is the angular velocity with which the axis of ω_2 separates from that of Ω_1 when the two systems of axes coincide, and may be represented by the symbol θ_3; then also the velocity with which the axis of ω_1 separates from that of Ω_2 at the instant when the two systems of axes coincide will be $-\theta_3$; and making θ_1, θ_2 the similar velocities of separation of the two other pairs of axes, we have

$$\left.\begin{aligned}\frac{d\Omega_1}{dt} &= \frac{d\omega_1}{dt} - \theta_3\omega_2 + \theta_2\omega_3\\ \frac{d\Omega_2}{dt} &= \frac{d\omega_2}{dt} - \theta_1\omega_3 + \theta_3\omega_1\\ \frac{d\Omega_3}{dt} &= \frac{d\omega_3}{dt} - \theta_2\omega_1 + \theta_1\omega_2\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots (7).$$

5. Although the above formulæ are interesting in a geometrical point of view, they will only be useful in simplifying the equations of motion of a rigid body, when the systems of axes are principal axes of the body. And there are two cases in which they may be used for that purpose.

6. First, when the body is such that the moments of inertia about two principal axes are equal, in that case all axes in the plane of these two will be principal axes, and if Ω_1, Ω_2 be the angular velocities about two principal axes fixed in the body, and ω_1, ω_2 the angular velocities about two principal axes moving in a given manner, the third axis being the same for both systems, then if Euler's axes fixed in the body coincide at the time t with the moveable axes, we must substitute in Euler's equations, for $\frac{d\Omega_1}{dt}$ the value $\frac{d\omega_1}{dt} - \theta_3\omega_2$, and for $\frac{d\Omega_2}{dt}$ the value $\frac{d\omega_2}{dt} + \theta_3\omega_1$.

θ_1 and θ_2 being both equal to zero, the equations of motion then become

$$\left.\begin{aligned}A\left(\frac{d\omega_1}{dt} - \theta_3\omega_2\right) + (C-A)\,\omega_2\omega_3 &= L\\ A\left(\frac{d\omega_2}{dt} + \theta_3\omega_1\right) - (C-A)\,\omega_1\omega_3 &= M\\ C\frac{d\omega_3}{dt} \qquad\qquad\qquad\qquad &= N\end{aligned}\right\}\ldots\ldots\ldots (8),$$

A being the moment of inertia about the axis of ω_1 or ω_2 and C about that of ω_3: and L, M, N the moments of the impressed forces about those axes.

7. Again, the equations (7) may be used in all their generality when the three principal moments of inertia are equal, for in that case all axes through the point are principal axes: and we may therefore choose that rectangular system which gives the simplest expressions for the moments of the forces about them. And if Ω_1, Ω_2, Ω_3 be about axes fixed in space, we have equations of motion

$$\left.\begin{aligned} A\left(\frac{d\omega_1}{dt} - \theta_3\omega_2 + \theta_2\omega_3\right) &= L \\ A\left(\frac{d\omega_2}{dt} - \theta_1\omega_3 + \theta_3\omega_1\right) &= M \\ A\left(\frac{d\omega_3}{dt} - \theta_2\omega_1 + \theta_1\omega_2\right) &= N \end{aligned}\right\} \quad \text{............ (9).}$$

And if x, y, z be the coordinates of a particle referred to these moving axes, it is evident from what precedes that the velocities parallel to them are

$$\left.\begin{aligned} u &= \frac{dx}{dt} - y\theta_3 + z\theta_2 \\ v &= \frac{dy}{dt} - z\theta_1 + x\theta_3 \\ w &= \frac{dz}{dt} - x\theta_2 + y\theta_1 \end{aligned}\right\} \quad \text{................ (10),}$$

and the accelerations are

$$\left.\begin{aligned} &\frac{du}{dt} - \theta_3 v + \theta_2 w \\ &\frac{dv}{dt} - \theta_1 w + \theta_3 u \\ &\frac{dw}{dt} - \theta_2 u + \theta_1 v \end{aligned}\right\} \quad \text{.................. (11).}$$

8. We shall now add a few problems for the purpose of illustrating the application of the preceding formulæ.

(1) A rigid body is moving about a fixed point under the action of given forces, to determine the pressure on the

point in terms of the angular velocities about the principal axes.

Let X', Y', Z', be the components of the action of the point on the body parallel to the principal axes, X, Y, Z the components of the impressed force on the element δm, and L, M, N the moments of the whole impressed forces about the principal axes, let h, k, l be the coordinates of the centre of gravity, u, v, w its component velocities, and m the mass of the body; then by (1)

$$u = -k\omega_3 + l\omega_2,$$
$$v = -l\omega_1 + h\omega_3,$$
$$w = -h\omega_2 + k\omega_1,$$

and $$m\left(\frac{du}{dt} - v\omega_3 + w\omega_2\right) = X' + \Sigma\delta m X,$$

that is

$$X' + \Sigma\delta m X = m\left\{l\frac{d\omega_2}{dt} - k\frac{d\omega_3}{dt} - h(\omega_2^2 + \omega_3^2) + \omega_1(k\omega_2 + l\omega_3)\right\},$$

and $$B\frac{d\omega_2}{dt} - (C - A)\,\omega_1\omega_3 = M,$$

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = N,$$

whence $$X' + \Sigma\delta m X = l\frac{M}{B} - \frac{kN}{C} + (C + B - A)\,\omega_1\left(\frac{l\omega_3}{B} + \frac{k\omega_2}{C}\right)$$
$$- h(\omega_2^2 + \omega_3^2),$$

whence X' is known, and similar expressions give Y' and Z'.

(2) A body is moving about a fixed point under the action of given forces, to find the whole force of constraint during the motion, on the part of the body cut off by a given surface in it.

Let ω_1, ω_2, ω_3 be the angular velocities of the body about its principal axes.

ω_1', ω_2', ω_3' the angular velocities about lines parallel to the principal axes of the part cut off, at its centre of gravity.

h, k, l the coordinates of the centre of gravity of the part cut off.

a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 the direction cosines of its principal axes referred to the principal axes of the whole body.

The force of constraint required can evidently be reduced to a force at the centre of gravity of the part cut off, and a couple.

Let X', Y', Z' be the components of this force parallel to the principal axes of the whole body: and X, Y, Z the forces impressed on the element $\delta m'$ of the part cut off.

Also let L', M', N' be the components of the couple of constraint, about the principal axes of the part cut off and L, M, N the similar components due to the impressed forces on the element $\delta m'$.

Then if m' = mass of the part cut off, the component velocities of its centre of gravity are

$$u = -k\omega_3 + l\omega_2,$$

$$v = -\ l\omega_1 + h\omega_3,$$

$$w = -h\omega_2 + k\omega_1,$$

and
$$m'\left(\frac{du}{dt} - v\omega_3 + w\omega_2\right) = X' + \Sigma\,\delta m' X,$$

or
$$m'\left\{-k\frac{d\omega_3}{dt} + l\frac{d\omega_2}{dt} - h(\omega_3^2 + \omega_2^2) + \omega_1(l\omega_3 + k\omega_2)\right\} = X' + \Sigma\delta m' X,$$

whence X' is found, and Y', Z' are found from two similar equations.

Again, if A', B', C' are the moments of inertia of the part cut off, we have, for the rotation of the part about its centre of gravity,

$$A'\frac{d\omega_1'}{dt} - (B' - C')\,\omega_2'\omega_3' = L' + \Sigma\,\delta m' L,$$

$$B'\frac{d\omega_2'}{dt} - (C' - A')\,\omega_3'\omega_1' = M' + \Sigma\,\delta m' M,$$

$$C'\frac{d\omega_3'}{dt} - (A' - B')\,\omega_1'\omega_2' = N' + \Sigma\,\delta m' N,$$

and
$$\omega_1' = a_1\omega_1 + b_1\omega_2 + c_1\omega_3, \text{ \&c.},$$

$$\frac{d\omega_1'}{dt} = a_1\frac{d\omega_1}{dt} + b_1\frac{d\omega_2}{dt} + c_1\frac{d\omega_3}{dt}\dots$$

from the above equations X', Y', Z' and L', M', N' may be found when ω_1, ω_2, ω_3 are known.

(3) A sphere rolls on the surface of another fixed sphere under the action of gravity, having been started with an

angular rotation about the line joining the centres of the two spheres, to determine the motion.

Let D (fig. 9) be the highest point of the fixed sphere, C the point of contact of the two spheres at the time t, CA CB arcs subtending a right angle at the centre of the sphere, and $\angle ACB$ a right angle, DX any fixed vertical arc, then we shall refer the motion to axes joining the centre of the fixed sphere to the points A, B, C on its surface. Let $\angle CDX = \phi$ and the angle subtended by CD at the centre $= \theta$.

$b =$ radius of fixed sphere, $a =$ radius of moving sphere.

X, Y, Z the components of the friction and pressure parallel to the three axes.

Then A approaches a fixed point coinciding with the instantaneous position of B with angular velocity $\frac{d\phi}{dt}\cos\theta$.

C separates from a fixed point at B with velocity $\frac{d\phi}{dt}\sin\theta$.

C moves from fixed point at A with angular velocity $\frac{d\theta}{dt}$, hence referring to equations (9) we see that

$$\theta_1 = \frac{d\phi}{dt}\sin\theta,\quad \theta_2 = -\frac{d\theta}{dt},\quad \text{and } \theta_3 = \frac{d\phi}{dt}\cos\theta.$$

And the equations for the motion of the sphere about its centre of gravity are

$$mk^2\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\phi}{dt}\cos\theta - \omega_3\frac{d\theta}{dt}\right) = Ya \ldots\ldots(1),$$

$$mk^2\left(\frac{d\omega_2}{dt} - \omega_3\frac{d\phi}{dt}\sin\theta + \omega_1\frac{d\phi}{dt}\cos\theta\right) = -Xa \ldots(2),$$

$$mk^2\left(\frac{d\omega_3}{dt} + \omega_1\frac{d\theta}{dt} + \omega_2\frac{d\phi}{dt}\sin\theta\right) = 0 \ldots\ldots(3).$$

And the coordinates of the centre of the sphere are

$$x = 0,\quad y = 0,\quad z = (a+b) = c \text{ say},$$

hence by (10) the velocities of its centre of gravity are

$$u = z\theta_2 = -c\frac{d\theta}{dt} \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$v = -c\theta_1 = -c\frac{d\phi}{dt}\sin\theta \ldots\ldots\ldots\ldots (5),$$

$$w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

And for the motion of the centre of the sphere we have, since $w=0$,

$$m\left(\frac{du}{dt}-v\frac{d\phi}{dt}\cos\theta\right)=X-mg\sin\theta\,\ldots\ldots\ldots(7),$$

$$m\left(\frac{dv}{dt}+u\frac{d\phi}{dt}\cos\theta\right)=Y\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$m\left(u\frac{d\theta}{dt}+v\frac{d\phi}{dt}\sin\theta\right)=Z-mg\cos\theta\,\ldots\ldots\ldots(9).$$

And from the condition of perfect rolling we have

$$u-a\omega_2=0;\ \text{therefore}\ \omega_2=-\frac{c}{a}\frac{d\theta}{dt}\,\ldots\ldots\ldots(10),$$

$$v+a\omega_1=0;\ \text{therefore}\ \omega_1=\frac{c}{a}\sin\theta\frac{d\phi}{dt}\,\ldots.(11),$$

hence
$$\omega_1\frac{d\theta}{dt}+\omega_2\sin\theta\frac{d\phi}{dt}=0,$$

whence (3) becomes
$$\frac{d\omega_3}{dt}=0;$$

therefore
$$\omega_3=\text{constant}=n.$$

Again, from (1) and (8) combined with (10) and (11),

$$mk^2\left(\frac{d\omega_1}{dt}-\omega_2\frac{d\phi}{dt}\cos\theta-n\frac{d\theta}{dt}\right)+ma^2\left(\frac{d\omega_1}{dt}-\omega_2\frac{d\phi}{dt}\cos\theta\right)=0;$$

therefore
$$(k^2+a^2)\left(\frac{d\omega_1}{dt}-\omega_2\frac{d\phi}{dt}\cos\theta\right)=k^2n\frac{d\theta}{dt},$$

and
$$\omega_2\frac{d\phi}{dt}\cos\theta=-\omega_1\cot\theta\frac{d\theta}{dt}\quad\text{and}\quad k^2=\tfrac{2}{5}a^2;$$

therefore
$$7\left(\frac{d\omega_1}{dt}\sin\theta+\omega_1\cos\theta\frac{d\theta}{dt}\right)=2n\sin\theta\frac{d\theta}{dt};$$

therefore
$$7\omega_1\sin\theta=C-2n\cos\theta,$$

to simplify the constants we shall suppose that initially θ is indefinitely small and $\omega_1=\omega_2=0$; therefore

$$7\omega_1\sin\theta=2n(1-\cos\theta);$$

therefore
$$\omega_1=\tfrac{2}{7}n\tan\tfrac{1}{2}\theta.$$

Again, eliminating X from (7) and (2) taking account of (10) and (11)

$$k^2\left(\frac{d\omega_2}{dt}+\omega_1\frac{d\phi}{dt}\cos\theta-n\frac{d\phi}{dt}\sin\theta\right)+a^2\left(\frac{d\omega_2}{dt}+\omega_1\frac{d\phi}{dt}\cos\theta\right)$$
$$=-ag\sin\theta,$$

or $$(k^2+a^2)\left(\frac{d\omega_2}{dt}+\omega_1\frac{d\phi}{dt}\cos\theta\right)-nk^2\frac{d\phi}{dt}\sin\theta=-ag\sin\theta,$$

and $$(k^2+a^2)\left(\frac{d\omega_1}{dt}-\omega_2\frac{d\phi}{dt}\cos\theta\right)-nk^2\frac{d\theta}{dt}=0;$$

therefore $$(k^2+a^2)\left(\omega_1\frac{d\omega_1}{dt}+\omega_2\frac{d\omega_2}{dt}\right)=gc\sin\theta\frac{d\theta}{dt};$$

therefore $$7a^2(\omega_1^2+\omega_2^2)=C_1-10gc\cos\theta$$
$$=10gc(1-\cos\theta),$$

and $$\omega_1=\frac{2n}{7}\tan\tfrac{1}{2}\theta;\quad \omega_2=-\frac{c}{a}\frac{d\theta}{dt};$$

therefore $$c^2\left.\frac{d\theta}{dt}\right|^2+\frac{4n^2a^2}{4g}\tan^2\tfrac{1}{2}\theta=\frac{20gc}{7}\sin^2\tfrac{1}{2}\theta,$$

whence $$t=\frac{7c}{2}\int\frac{d\theta\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\sqrt{(35gc\cos^2\frac{1}{2}\theta-a^2n^2)}},$$

an expression which can easily be integrated, and thus the time of coming from any one position to any other may be found

Again, $$\frac{d\phi}{dt}=\frac{a\omega_1}{c\sin\theta}=\frac{an}{7c}\sec^2\tfrac{1}{2}\theta;$$

therefore $$\frac{d\phi}{d\theta}=\frac{an}{\sin\theta\sqrt{(35gc\cos^2\frac{1}{2}\theta-a^2n^2}},$$

whence the path described by the point of contact may be found.

Again, to find where the spheres will separate, we have from (9)

$$Z=mg\cos\theta+m\left(u\frac{d\theta}{dt}+v\frac{d\phi}{dt}\sin\theta\right)$$
$$=mg\cos\theta-m\frac{a^2}{c}(\omega_2^2+\omega_1^2)$$
$$=mg\{\cos\theta-\tfrac{10}{7}(1-\cos\theta)\}$$
$$=\frac{mg}{7}(17\cos\theta-10),$$

and therefore they will separate when $\cos\theta=\frac{10}{17}$.

(4) To determine the motion of a sphere on the surface of a perfectly rough cone having its axis vertical.

Through the centre of the sphere draw a line parallel to the generating line which touches the sphere, meeting the axis of the cone in the point O. Take O as origin, the line so drawn as axis of x, and the axis of z perpendicular to it in the plane through the axis of the cone, the axis of y perpendicular to the plane xz.

Let $\frac{1}{2}\pi - \alpha$ be half the vertical angle of the cone, and as in the last example let A, B, C, D be the points in which the coordinate axes and axis of the cone meet the surface of a sphere whose centre is at O; DX a fixed vertical circle.

$\angle CDX = \phi$, a = radius of sphere.

X, Y, Z the components of the force at the point of contact.

The coordinates of the centre are x, 0, 0. Also

$$\theta_1 = \frac{d\phi}{dt}\sin\alpha, \quad \theta_2 = 0, \quad \theta_3 = \frac{d\phi}{dt}\cos\alpha;$$

therefore, for the rotation about the centre of gravity, we have by (9)

$$mk^2\left(\frac{d\omega_1}{dt} - \omega_2\frac{d\phi}{dt}\cos\alpha\right) = Ya \ldots (1),$$

$$mk^2\left(\frac{d\omega_2}{dt} - \omega_3\frac{d\phi}{dt}\sin\alpha + \omega_1\frac{d\phi}{dt}\cos\alpha\right) = -Xa \ldots (2),$$

$$mk^2\left(\frac{d\omega_3}{dt} + \omega_2\frac{d\phi}{dt}\sin\alpha\right) = 0 \ldots\ldots (3),$$

and if u, v, w be the component velocities of the centre of the sphere by (10),

$$u = \frac{dx}{dt}, \quad v = x\frac{d\phi}{dt}\cos\alpha, \quad w = 0,$$

for the motion of translation

$$m\left(\frac{du}{dt} - v\frac{d\phi}{dt}\cos\alpha\right) = X - mg\sin\alpha \ldots\ldots\ldots (4),$$

$$m\left(\frac{dv}{dt} + u\frac{d\phi}{dt}\cos\alpha\right) = Y \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$mv\frac{d\phi}{dt}\sin\alpha = Z - mg\cos\alpha \ldots\ldots\ldots\ldots (6),$$

and for rolling

$$u - a\omega_2 = 0, \quad v + a\omega_1 = 0 \ldots\ldots\ldots\ldots\ldots (7).$$

Eliminate Y from (1) and (5) and take account of (7); therefore

$$\frac{dv}{dt} + u\frac{d\phi}{dt}\cos\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and $$v = x\frac{d\phi}{dt}\cos\alpha;\ \text{therefore}\ x\frac{dv}{dt} + v\frac{dx}{dt} = 0;$$

therefore $$vx = \text{constant} = C \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

from (9) and (3)

$$a\frac{d\omega_3}{dt} = \tan\alpha\frac{dv}{dt};\ \text{therefore}\ a\omega_3 = (v + v_0)\tan\alpha \ldots(10).$$

Eliminate X from (2) and (5), substitute for $\frac{d\phi}{dt}$ from (8) and reduce, then

$$(k^2 + a^2)u\frac{du}{dt} + \frac{k^2 + a^2\cos^2\alpha}{\cos\alpha}v\frac{dv}{dt} + k^2v_0\frac{\sin^2\alpha}{\cos\alpha}\frac{dv}{dt} = -a^2g\cos\alpha\frac{dx}{dt};$$

therefore

$$(k^2 + a^2)\left(\frac{dx}{dt}\right)^2 + \frac{k^2 + a^2\cos^2\alpha}{\cos\alpha}\frac{c^2}{x^2} + \frac{2k^2v_0C\sin^2\alpha}{x\cos\alpha} = -2a^2gx + C',$$

whence t may be found in terms of x by integrating an expression of the form

$$\frac{dt}{dx} = \frac{x}{\sqrt{(A + Bx + Cx^2 + Dx^3)}},$$

and $$\frac{d\phi}{dt} = \frac{v}{x\cos\alpha} = \frac{c}{x^2\cos\alpha};\ \therefore \frac{d\phi}{dx} = \frac{c\sec\alpha}{x\sqrt{(A+Bx+Cx^2+Dx^3)}},$$

from which the polar equation, to the projection on a horizontal plane of the path of the centre of the sphere, may be found.

April, 1858.

PROPOSITION IN THE PLANETARY THEORY.

By the Rev. PERCIVAL FROST, M.A.

THE object of the following investigation is to obtain a relation between the mean distances of the instantaneous ellipses to the first order of the disturbing forces when the orbits of two planets m and m' are disturbed by their mutual influences.

I have communicated the investigation because I am not aware that the result has been previously obtained: and I have employed the notation which has been used in former numbers of this *Journal*, with some symbols which will be explained *en passant*.

The equation for determining the variation of a, is

$$\frac{\mu}{a_1^2}\frac{da_1}{dt}=2\frac{dR}{d(t)},$$

which may be written

$$\frac{\mu}{m'}\frac{d}{dt}\left(\frac{1}{a_1}\right)=-2\frac{d}{d(t)}\left(\frac{1}{\rho}-\frac{r\cos\omega}{r'^2}\right).$$

Let $\phi=N\alpha$ (fig. 10) be the angle between the perihelion of m's orbit and the line of intersection of the planes of the instantaneous orbits of m and m'. $v=$ the true anomaly at the time t; and let ϕ', v' be corresponding quantities for m'. γ the inclination of the orbits.

The spherical triangle Nmm' gives the equation

$$\cos\omega=\cos(v+\phi)\cos(v'+\phi')+\sin(v+\phi)\sin(v'+\phi')\cos\gamma.$$

If δu denote the change of any function u due to a change of position of m in the time dt, $\delta' u$ that due to a change of position of m',

$$hdt=r^2\delta v \text{ and } h'dt=r'^2\delta' v',$$

$$\frac{h^2}{\mu r}=1+e\cos v;$$

therefore $$\frac{h^2\delta r}{\mu r^2}=e\sin v\,\delta v;$$

therefore $$\delta r=\frac{e\mu\sin v}{h}dt=\frac{e\mu\sin v}{hh'}r'^2\delta' v',$$

and $$\frac{r\delta v}{r'^2}=\frac{h}{h'r}\delta' v'=\frac{\mu}{hh'}(1+e\cos v)\,\delta' v',$$

$$\delta\left(\frac{r\cos\omega}{r'^2}\right)=\frac{\delta r\cos\omega+r\dfrac{d\cos\omega}{dv}\delta v}{r'^2}$$

$$=\frac{\mu}{hh'}\left\{e\sin v\cos\omega+(1+e\cos v)\frac{d\cos\omega}{dv}\right\}\delta' v'$$

$$=\frac{\mu}{hh'}[e\sin v\{\cos(v+\phi)\cos(v'+\phi')+\sin(v+\phi)\sin(v'+\phi')\cos\gamma\}$$
$$+(1+e\cos v)\{-\sin(v+\phi)\cos(v'+\phi')$$
$$+\cos(v+\phi)\sin(v'+\phi')\cos\gamma\}]\,\delta v$$

$$= \frac{\mu}{hh'} [\sin(v' - v + \phi' - \phi) \cos^2 \tfrac{1}{2}\gamma - \sin(v' + v + \phi' + \phi) \sin^2 \tfrac{1}{2}\gamma$$

$$- e\{\sin\phi \cos(v' + \phi') - \cos\phi \sin(v' + \phi') \cos\gamma\}] \, \delta' v'$$

$$= \frac{\mu}{hh'} [\{\sin(v' - v + \phi' - \phi) \cos^2 \tfrac{1}{2}\gamma - \sin(v' + v + \phi' + \phi) \sin^2 \tfrac{1}{2}\gamma\} \, \delta' v'$$

$$- e\delta' \{\cos(v' + \phi' - \phi) \cos^2 \tfrac{1}{2}\gamma - \cos(v' + \phi' + \phi) \sin^2 \tfrac{1}{2}\gamma\}]$$

$$= \frac{\mu}{hh'} \{P' \delta' v' - \delta' Q'\}.$$

Now $$\frac{du}{dt} = \frac{(\delta + \delta') u}{dt},$$

and $$\frac{\mu}{m'} \frac{d}{dt} \left(\frac{1}{a_1}\right) = - \frac{2\delta \left(\frac{1}{\rho}\right)}{dt} + \frac{2\mu}{hh'} \left(P' \frac{\delta' v'}{dt} - \frac{\delta' Q'}{dt}\right),$$

$$\frac{\mu}{m} \frac{d}{dt} \left(\frac{1}{a_1'}\right) = - \frac{2\delta' \frac{1}{\rho}}{dt} + \frac{2\mu}{hh'} \left(P \frac{\delta v}{dt} - \frac{\delta Q}{dt}\right);$$

therefore $$\frac{d}{dt} \left(\frac{\mu}{m' a_1} + \frac{\mu}{m a_1'}\right) = - \frac{2d}{dt} \left(\frac{1}{\rho}\right) + \frac{2\mu}{hh'} \left(P' \frac{dv'}{dt} + P \frac{dv}{dt}\right)$$

$$- \frac{2\mu}{hh'} \frac{d}{dt} (Q + Q').$$

Since Q, and Q' involve only v and v' respectively, also

$$P' \frac{dv'}{dt} + P \frac{dv}{dt}$$

$$= \{\sin(v' - v + \phi' - \phi) \cos^2 \tfrac{1}{2}\gamma - \sin(v' + v + \phi' + \phi) \sin^2 \tfrac{1}{2}\gamma\} \frac{dv'}{dt}$$

$$+ \{\sin(v - v' + \phi - \phi') \cos^2 \tfrac{1}{2}\gamma - \sin(v + v' + \phi + \phi') \sin^2 \tfrac{1}{2}\gamma\} \frac{dv}{dt}$$

$$= - \frac{d}{dt} \{\cos(v' - v + \phi' - \phi) \cos^2 \tfrac{1}{2}\gamma - \cos(v' + v + \phi' + \phi) \sin^2 \tfrac{1}{2}\gamma\};$$

therefore $$\frac{m}{a_1} + \frac{m'}{a_1'} = C - \frac{2mm'}{\mu\rho} - \frac{2mm'}{hh'} [\cos(v' - v + \phi' - \phi) \cos^2 \tfrac{1}{2}\gamma$$

$$- \cos(v' + v + \phi' + \phi) \sin^2 \tfrac{1}{2}\gamma + \{e \cos(v' + \phi' - \phi) + e' \cos(v + \phi - \phi')\} \cos^2 \tfrac{1}{2}\gamma$$

$$- \{e \cos(v' + \phi' + \phi) + e' \cos(v + \phi + \phi')\} \sin^2 \tfrac{1}{2}\gamma],$$

whence also we obtain

$$m \sqrt{(a_1)}\, n_1 + m' \sqrt{(a_1')}\, n_1' = \frac{1}{\sqrt{(\mu)}} \left(\frac{m}{a_1} + \frac{m'}{a_1'}\right).$$

It is easily seen that if mN, $m'N$ be produced to MA and $M'A'$, so that mM, $m'M'$, aA, $a'A'$ are quadrants,

$$\frac{m}{a}+\frac{m'}{a'}=C-\frac{2mm'}{\mu\rho}-\frac{2mm'}{hh'}\{\cos MM'+e\cos AM'+e'\cos A'M\}$$

which last factor becomes, if $\gamma=0$,

$$\cos mm'+e\cos am'+e'\cos a'm.$$

NOTE ON THE RADIUS OF ABSOLUTE CURVATURE AT ANY POINT OF A CURVE.

CONSIDER the curve as the limit of an equilateral polygon, let BA, AC be two consecutive sides of the polygon, join BC and bisect it in O, then if ρ be the radius of absolute curvature,

$$\rho=\tfrac{1}{2}\text{ limit }\frac{AB^2}{AO}.$$

Let $BA=AC=\delta s$, and let x, y, z be the coordinates of A. Then those of B, C will respectively be

$$x-\frac{dx}{ds}\delta s+\tfrac{1}{2}\frac{d^2x}{ds^2}\delta s^2+\ldots,\quad x+\frac{dx}{ds}\delta s+\tfrac{1}{2}\frac{d^2x}{ds^2}\delta s^2+\ldots.$$

Hence those of O are

$$x+\tfrac{1}{2}\frac{d^2x}{ds^2}\delta s^2+\ldots,\quad y+\tfrac{1}{2}\frac{d^2y}{ds^2}\delta s^2+\ldots,\quad z+\tfrac{1}{2}\frac{d^2z}{ds^2}\delta s^2+\ldots;$$

therefore $\rho=\tfrac{1}{2}\text{ limit }\dfrac{\delta s^2}{\tfrac{1}{2}\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2+\ldots\right\}^{\frac{1}{2}}\delta s^2}$

$$=\frac{1}{\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}}},$$

the known expression for the magnitude of the radius of absolute curvature.

The equations to its direction, from the same construction, will be

$$\frac{\xi-x}{\frac{d^2x}{ds^2}}=\frac{\eta-y}{\frac{d^2y}{ds^2}}=\frac{\zeta-z}{\frac{d^2z}{ds^2}}.$$

N. M. F.

NOTE ON AREAS AND VOLUMES IN TRILINEAR AND QUADRIPLANAR COORDINATES.

By G. M. Slesser, B.A., Queens' College.

In the February Number of this *Journal*, Mr. Ferrers has determined the area of the ellipse represented by the general trilinear equation of the second degree; it is the object of the present note, to shew how the areas and volumes of closed curves, or surfaces, whose trilinear or quadriplanar equations are given, may be found by integration.

Let the position of a point P, as in Mr. Ferrers' paper, be given by the ratios α, β, γ of the triangles PBC, PCA, PAB, respectively, to the triangle of reference ABC, so that $\alpha+\beta+\gamma=1$ identically; let x, y be rectangular coordinates of the same point, then since α, β are linear functions of x, y, if we transform the double integral $\iint dxdy$, so that α, β may be independent variables, we shall have $\iint dxdy = C\iint d\alpha d\beta$, where C is some constant; and integrating $\iint dxdy$ over the whole triangle of reference we have $\iint dxdy = S$, the area of the triangle, and in integrating $\iint d\alpha d\beta$ over the triangle, the limits of β are from $\beta=0$ to $\gamma=0$, or $\beta=1-\alpha$.

And the limits of α are 1, 0.

And $$\int_0^1\int_0^{1-\alpha} d\alpha d\beta = \tfrac{1}{2};$$

therefore $$S=\tfrac{1}{2}C \text{ or } C=2S;$$

therefore the area of any closed curve $\phi(\alpha, \beta, \gamma)=0$ is

$$2S\iint d\alpha d\beta,$$

the limits of integration being given by the equation

$$\phi(\alpha, \beta, 1-\alpha-\beta)=0.$$

Again, if the position of a point P be determined by the ratios α, β, γ, δ of the tetrahedra, whose vertex is P and bases the four faces of the tetrahedron of reference, to the tetrahedron of reference, so that $\alpha+\beta+\gamma+\delta=1$ identically, we have as before

$$\iiint dxdydz = C\iiint d\alpha d\beta d\gamma,$$

and $\iiint dxdydz$ for the tetrahedron $=V$, the volume of the tetrahedron.

Also to evaluate $\iiint d\alpha d\beta d\gamma$ for that tetrahedron the limits of γ are from $\delta = 0$, to $\gamma = 0$, or $\gamma = 1 - \alpha - \beta$, and $\gamma = 0$

$$\text{limits of } \beta \ldots \text{ are } \beta = 1 - \alpha,\ \beta = 0,$$

$$\ldots\ldots\ldots\ \alpha\ \ldots\ldots\ \alpha = 1, \qquad \alpha = 0;$$

therefore $\iiint d\alpha d\beta d\gamma$ for the tetrahedron $= \frac{1}{6}$; therefore

$$V = \tfrac{1}{6} C;\ \text{therefore } C = 6V.$$

Hence the volume of any closed surface $\phi(\alpha, \beta, \gamma, \delta) = 0$ is

$$6V \iiint d\alpha d\beta d\gamma,$$

the limits of integration being given by the equation

$$\phi(\alpha, \beta, \gamma, 1 - \alpha - \beta - \gamma) = 0.$$

THE PLANETARY THEORY.

(Continued from p. 107.)

By the Rev. PERCIVAL FROST, M.A.

35. It has now been shewn that the disturbing function can be separated into two distinct portions, differing essentially in their characters; that one of these portions depends only on the elements of the orbits of the disturbing and disturbed planets and not on the relative positions of the planets in their orbits; and that the other portion consist of a series of periodic terms which do depend on this relative position, each of which passes through all its values, when the planets have completed definite numbers of revolutions. The first portion has been denoted by the symbol F, and has been calculated as far as terms of the second order in small quantities, at least for the cases in which the mean motions of the planets are not exactly commensurable, for it has not been considered necessary to discuss at length the case of exact commensurability, since no such case has yet arisen in our system, although a very near approach to it is made in the case of Uranus and Neptune.

However, it will be well to shew, in a few words, how terms would be abstracted from the periodic portions and become a part of F.

Thus, if n, n' were in the ratio of two whole numbers m, m' or $nm' - m'n = 0$ there would arise terms in the developement of R originating in the product of such factors as

$$\cos\{m(n't + \epsilon' - nt - \epsilon) + \alpha\} \cos\{(m + m')(nt + \epsilon) + \beta\},$$

which would be of the form

$$A \cos\{(mn' - m'n)t + \gamma\} = A \cos\gamma,$$

which would appear as an additional term in F.

We may observe also that this exact commensurability would only remain for a short time, being in an unstable state, and on the recommencement of the incommensurability the methods adopted will be again brought into play.

The type of the second set of terms, called periodic, is $Q \cos(pnt - qn't + \alpha)$ in which Q is of the order $p \sim q$ in small quantities, and this term will pass through all its values when t has been increased by $\frac{2\pi}{pn \sim qn'}$, in which time the planets m and m' will have revolved for $\frac{n}{pn \sim qn'}$ and $\frac{n'}{pn \sim qn'}$ of their respective periods; or, if P, P' be the sidereal periods of the planets, the period of this inequality is $\frac{PP'}{pP' \sim qP}$, which will be very large if $P : P' = p : q$ nearly.

On the Secular Variations of the Elements.

36. We shall at this point occupy ourselves with the examination of the average effect of the action of a disturbing planet upon the elements of a disturbed planet's orbit in the course of a very large number of revolutions.

In order to do this we shall employ in the equations which give the variations of the elements the mean value of the disturbing function, which is the non-periodic portion denoted by F.

The variations so found are the secular variations of the elements, and an orbit constructed at any time with the elements obtained by this process, will represent the average of all the disturbances caused up to that time by the action of the disturbing planet; and such an orbit would differ as well from the instantaneous orbit as from the orbit absolutely described by the planet under the action of the disturbing force.

37. For all the ordinary purposes of Astronomy, it is sufficient to approximate to the values of the elements by considering those elements, which appear in the equations for the secular variations, as constant and equal to their values at the epoch chosen for the origin of the time.

Thus, if e_1 be the element in question, we have, by Maclaurin's theorem,

$$e_1 = e_0 + \left(\frac{de_1}{dt}\right)_0 t + \left(\frac{d^2e_1}{dt^2}\right)_0 \frac{t^2}{2} + \ldots\ldots$$

$$\frac{de_1}{dt} = \text{a known function of the elements} = U_e$$

$$\frac{d^2e_1}{dt^2} = \frac{dU_e}{da_1}\frac{da_1}{dt} + \frac{dU_e}{de_1}\frac{de_1}{dt} + \ldots\ldots$$

$$= \frac{dU_e}{da_1} U_a + \frac{dU_e}{de_1} U_e + \ldots\ldots$$

&c. = &c.

whence $\frac{de_1}{dt}$, $\frac{d^2e_1}{dt^2}$, can be calculated for the origin of the time, and e_1 can be found for any degree of accuracy, and for a century before or after the epoch considered, the terms after t^2 might be neglected, since $\left(\frac{d^3e_1}{dt^3}\right)_0$ is exceedingly small being of the third order of the disturbing force, $\left(\frac{de_1}{dt}\right)_0$ is the annual variation of e_1, and can be calculated without difficulty.

Secular Variations of the mean Distance.

38. Among all the elements of the instantaneous ellipse, that which claims our attention in the first place is the major axis of the orbit, for upon this element depend the periodic time and mean motion of the planet as well as its distance from the Sun.

We will determine then in the first place whether, neglecting the squares of the disturbing forces, the mean distance of the disturbed planet's orbit from the Sun will be permanently changed, irrespective of the changes due to the eccentricity of the orbit.

The equation, which determines the variations of a_1, is

$$\frac{da_1}{dt} = \frac{2n_1a_1^2}{\mu}\frac{dR}{d\varepsilon_1}.$$

The secular variation of a_1 is found by writing F for R, and since

$$\frac{dF}{d\varepsilon_1}=0;$$

therefore $$\frac{da_1}{dt}=0,$$

or a_1 has no secular variation; hence, the major axis remains unaltered.

Secular Variation of the mean Motion.

39. Before proceeding to the solution of the equation which gives the secular variation of the mean motion, we shall say a few words on its measure.

In the undisturbed elliptic motion of a planet about the Sun, the coordinates of the planet are expressed in terms of $nt+\varepsilon$ and series of sines and cosines of $nt+\varepsilon-\varpi$ and its multiples.

$nt+\varepsilon$ is the mean longitude, n the mean motion, and ε the mean longitude at the commencement of the epoch.

But in the orbit of the disturbed planet the longitude of the epoch must be so adjusted that the true longitude and distance, at the time t, shall be given by the elliptic formulæ involving $nt+\varepsilon$, the eccentricity and longitude of perihelion in the instantaneous orbit constructed for that time.

Now, since we can approximate as near as we please to the real orbit of the disturbed planet, by dividing the time t into a large number of small intervals, constructing the instantaneous ellipses for the commencement of each interval, and supposing the body to move during each interval as in the corresponding instantaneous ellipse, the orbit will then be obtained by increasing the number and diminishing the magnitude of the intervals *ad libitum*.

Hence we are at liberty to adopt either of the two following methods of estimating the mean motion during the time t, suppose ζ.

I. We may consider it as the mean angular velocity in the instantaneous ellipse multiplied by the time or $\zeta=n_1t$.

II. We may consider it the mean motion during the time t in the disturbed orbit, which is the same as the limit of the mean motions in the series of ellipses corresponding to the successive small intervals of time from the commencement of the epoch to the time t. In this case $\zeta=\int n_1dt$.

We shall see the advantage derived from the adoption of the second method with regard to the adjustment of the value of ϵ, if we recur to Art. 18, in which it is shewn that

$$\frac{d(\zeta+\epsilon_1)}{dt}=n_1-\frac{2n_1a_1^2}{\mu}\frac{dR}{da_1}+\ldots\ldots,$$

where $\frac{dR}{da_1}$ is the differential coefficient of R considering n_1 as independent of a_1, ζ being the mean motion in the time t.

In the first method of considering ζ, viz.

$$\zeta=n_1t,$$

and
$$\frac{d\epsilon_1}{dt}=-t\frac{dn_1}{dt}-\frac{2n_1a_1^2}{\mu}\frac{dR}{da_1}+\ldots\ldots,$$

and
$$-\frac{dn_1}{dt}t=-\frac{dn_1}{da_1}\frac{da_1}{dt}t=-\frac{dn_1}{da_1}\frac{2n_1a_1^2}{\mu}\frac{dR}{d(n_1t+\epsilon_1)}t$$
$$=-\frac{2na_1^2}{\mu}\frac{dR}{dn_1}\frac{dn_1}{da_1};$$

therefore
$$\frac{d\epsilon_1}{dt}=-\frac{2n_1a_1^2}{\mu}\left(\frac{dR}{da_1}+\frac{dR}{dn_1}\frac{dn_1}{da_1}\right)+\ldots\ldots,$$

where $\frac{dR}{da_1}+\frac{dR}{dn_1}\frac{dn_1}{da_1}$ denotes the complete differential coefficient of R, considering n_1 a function of a_1 by the equation $n_1^2a_1^3=\mu$.

In the second method of considering ζ in which $\zeta=\int n_1dt$,

$$\frac{d\epsilon_1}{dt}=-\frac{2n_1a_1^2}{\mu}\frac{dR}{da_1}+\ldots\ldots.$$

The second method has therefore the advantage of making the terms which depend on the time explicitly disappear and the formulæ in the instantaneous ellipse will be applicable by writing $\int n_1dt+\epsilon_1$ for $n_1t+\epsilon_1$.

40. Adopting therefore this method, since

$$\frac{2dn_1}{n_1dt}+\frac{3da_1}{a_1dt}=0,$$

$$\frac{dn_1}{dt}=-\frac{3n_1}{2a_1}\frac{2n_1a_1^2}{\mu}\frac{dR}{d\epsilon_1};$$

therefore
$$\zeta=\int n_1dt=-\iint\frac{3n_1^2a_1}{\mu}\frac{dR}{d\epsilon_1}dt^2,$$

and since $\frac{dF}{d\varepsilon_1} = 0$ the mean motion in the time t has no secular variation.

Hence, the mean motion and the major axis have none but periodic variations, or their variations depend only on the configuration of the bodies of the system.

On the Secular Variation of the Eccentricity and Longitude of Perihelion.

41. The general equations for determining the variations of the eccentricity and longitude of perihelion are

$$\frac{de_1}{dt} = -\frac{n_1 a_1 \sqrt{(1-e_1^2)}}{\mu e_1}\left(\frac{dR}{d\varepsilon_1} + \frac{dR}{d\varpi_1}\right) + \frac{n_1 a_1 (1-e_1^2)}{\mu e_1}\frac{dR}{d\varepsilon_1},$$

and $$\frac{d\varpi_1}{dt} = \frac{n_1 a_1 \sqrt{(1-e_1^2)}}{\mu e_1}\frac{dR}{de_1} + \frac{n_1 a_1 \tan\frac{1}{2}i_1}{\mu\sqrt{(1-e_1^2)}}\frac{dR}{di_1}.$$

If in these equations we write F for R, and observe that

$$\frac{dF}{d\varepsilon_1} = 0,$$

$$\frac{dF}{d\varpi_1} = -\frac{m' a_1 a_1'}{4} D_2 e_1 e_1' \sin(\varpi_1' - \varpi),$$

$$\frac{dF}{de_1} = \frac{m' a_1 a_1'}{4}\{D_1 e_1 - D_2 e_1' \cos(\varpi_1' - \varpi)\},$$

$$\frac{dF}{di_1} = -\frac{m' a_1 a_1'}{4} D_1 \{\tan i_1 - \tan i_1' \cos(\Omega_1' - \Omega)\},$$

we have, preserving only terms of the first order,

$$\frac{de_1}{dt} = \frac{m' n a^2 a'}{4\mu} D_2 e_1' \sin(\varpi_1' - \varpi),$$

$$e_1 \frac{d\varpi_1}{dt} = \frac{m' n a^2 a'}{4\mu}\{D_1 e_1 - D_2 e_1' \cos(\varpi_1' - \varpi)\},$$

and similar expressions for $\frac{de_1'}{dt}$ and $e_1' \frac{d\varpi_1'}{dt}$.

For $m\sqrt{(a)}$ and $m'\sqrt{(a')}$ write γ and γ', and $na^{\frac{3}{2}} = \sqrt{(\mu)}$; therefore

$$\frac{de_1}{dt} = \frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\,\gamma' D_2 e_1' \sin(\varpi_1' - \varpi_1),$$

$$e_1 \frac{d\varpi_1}{dt} = \frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\,\gamma'\{D_1 e_1 - D_2 e_1' \cos(\varpi_1' - \varpi)\}.$$

Let $$h = e_1 \sin \varpi_1, \quad h' = e_1' \sin \varpi_1',$$
$$k = e_1 \cos \varpi_1, \quad k' = e_1' \cos \varpi_1';$$

therefore $$\frac{dh}{dt} = \frac{de_1}{dt} \sin \varpi_1 + e_1 \frac{d\varpi_1}{dt} \cos \varpi_1$$
$$= \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} \gamma' (D_1 k - D_2 k'),$$
$$\frac{dk}{dt} = \frac{de_1}{dt} \cos \varpi_1 - e_1 \frac{d\varpi_1}{dt} \sin \varpi_1$$
$$= \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} \gamma' (- D_1 h + D_2 h').$$

Similarly $$\frac{dh'}{dt} = \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} \gamma (D_1 k' - D_2 k),$$

and $$\frac{dk'}{dt} = \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} \gamma (- D_1 h' + D_2 h).$$

It is easily seen, from the form of these equations, that particular integrals are

$$h = N \sin(gt + \alpha), \quad h' = N' \sin(gt + \alpha),$$
$$k = N \cos(gt + \alpha). \quad k' = N' \cos(gt + \alpha);$$

therefore substituting in either of the first two equations

$$gN = \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} \gamma' (D_1 N - D_2 N'),$$

and from the third or fourth

$$gN' = \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} \gamma (D_1 N' - D_2 N);$$

therefore

$$\left\{g - \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} D_1 \gamma'\right\} \left\{g - \tfrac{1}{4} \sqrt{\left(\frac{aa'}{\mu}\right)} D_1 \gamma\right\} = \frac{1}{16} \frac{aa'}{\mu} D_2^2 \gamma\gamma';$$

therefore $$g = \tfrac{1}{8} \sqrt{\left(\frac{aa'}{\mu}\right)} [(\gamma + \gamma') D_1 \pm \sqrt{\{(\gamma' - \gamma)^2 D_1^2 + 4\gamma\gamma' D_2^2\}}].$$

If g_1, g_2 be these two roots, $\alpha_1 \alpha_2$ and $N_1 N_2$ the corresponding values of α and N,

$$h = N_1 \sin(g_1 t + \alpha_1) + N_2 \sin(g_2 t + \alpha_2),$$
$$k = N_1 \cos(g_1 t + \alpha_1) + N_2 \cos(g_2 t + \alpha_2),$$

and similarly for h', k'.

And if e, ϖ, e', ϖ' be the values of e_1 and ϖ_1, &c. for the epoch from which t is reckoned, as 1850, the constants can be completely determined from the equation

$$e\cos\varpi = N_1 \sin\alpha_1 + N_2 \sin\alpha_2,$$
$$e\cos\varpi = N_1 \cos\alpha_1 + N_2 \cos\alpha_2,$$
$$e'\sin\varpi' = N_1' \cos\alpha_1 + N_2' \sin\alpha_2,$$
$$e'\cos\varpi' = N_1' \cos\alpha_1 + N_2' \cos\alpha_2,$$

$$\frac{N_1'}{N_1} = \frac{\frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1 - g_1}{\frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_2},$$

$$\frac{N_2'}{N_2} = \frac{\frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1 - g_2}{\frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_2}.$$

When these constants have been determined, we obtain

$$e_1^2 = h^2 + k^2 = N_1^2 + 2N_1N_2\cos\{(g_2 - g_1)t + \alpha_2 - \alpha_1\} + N_2^2,$$

$$\tan\varpi_1 = \frac{h}{k} = \frac{N_1 \sin(g_1 t + \alpha_1) + N_2 \sin(g_2 t + \alpha_2)}{N_1 \cos(g_1 t + \alpha_1) + N_2 \cos(g_2 t + \alpha_2)}.$$

Interpretation of the Expression for the Eccentricity.

42. Since $e_1^2 = N_1^2 + 2N_1N_2\cos\{(g_1 - g_2)t + \alpha_1 - \alpha_2\} + N_2^2$

$$= (N_1 + N_2)^2 \cos^2\left(\frac{g_1 - g_2}{2}t + \frac{\alpha_1 - \alpha_2}{2}\right)$$
$$+ (N_1 - N_2)^2 \sin^2\left(\frac{g_1 - g_2}{2}t + \frac{\alpha_1 - \alpha_2}{2}\right);$$

therefore e_1 varies between the limits $N_1 \sim N_2$ and $N_1 + N_2$ passing through all its values in the time $\dfrac{360 \times 60 \times 60}{g_1 \sim g_2}$ years, if g_1, g_2 be estimated in seconds, and the unit of time be one year.

Now, since $\dfrac{m'}{\mu}$ is a very small fraction for any body in our system, g_1 and g_2 are very small angles; therefore the period of the eccentricities is very great.

43. It is easy to see the following construction for the eccentricity at any time, derived from the above.

Let an ellipse APa be constructed, the reciprocals of whose semiaxes are $N_1 + N_2$ and $N_1 \sim N_2$, and let a point describe the circle AQa on the major axis as diameter, in $\frac{4\pi}{g_1 \sim g_2}$ years, commencing at Q_0, where $ACQ_0 = \frac{\alpha_1 - \alpha_2}{g_1 - g_2}$ at the epoch, which is the origin of the time. From its position at the time t, let QPM be drawn perpendicular to the major axis, meeting the ellipse in P, then CP is the reciprocal of the eccentricity.

Geometrical representation of the Secular Variations of e_1 *and* ϖ_1.

44. With the Sun's center (fig. 11) as center and radius $2aN_1$, let a circle be described, and let a point P describe this circle with uniform angular velocity g_1, commencing at a point A when $t=0$, so that $AS\Upsilon$ the longitude of $A = \alpha$, and $g_1 t + \alpha_1$ is the longitude of P. With P as center describe a circle whose radius is $2aN_2$, and let a point H describe this circle with uniform velocity g_2, commencing from B, so that PB is inclined at an angle α_2 to $S\Upsilon$, or $g_2 t + \alpha_2$ is the longitude of H.

Then it is easily seen that H is the upper focus of the ellipse at the time t, if $N_1 N_2$ be positive,

for $SH \cos HS\Upsilon$

$$= 2a \{N_1 \cos(g_1 t + \alpha_1) + N_2 \cos(g_2 t + \alpha_2)\} = 2ae \cos\varpi_1,$$

and $SH \sin HS\Upsilon$

$$= 2ae_1 \sin\varpi_1;$$

therefore $\qquad SH = 2ae_1, \quad$ and $HS\Upsilon = \varpi_1$.

If N_1 be positive and N_2 negative, produce HS to the circumference in H', then H' is the upper focus, and $H'S\Upsilon = \varpi_1$.

The figure is constructed for the case of Jupiter's orbit as disturbed by Saturn, in which we shall assume

$$e_1 \cos\varpi_1 = 0.044 \cos(g_1 t + \alpha_1) - 0.016 \cos(g_2 t + \alpha_2),$$

$$e_1 \sin\varpi_1 = 0.044 \sin(g_1 t + \alpha_1) - 0.016 \sin(g_2 t + \alpha_2),$$

$$g_1 = 3''.7, \quad \text{and } \alpha_1 = 27°.21' \quad \text{nearly,}$$

$$g_2 = 22''.4, \quad \text{and } \alpha_2 = 126°.44'$$

so that the small circle is described with a little less than six times the velocity of the large circle.

On the Secular Motion of the Apsidal Line.

45. Since

$$\tan \varpi_1 = \frac{N_1 \sin(g_1 t + \alpha_1) + N_2 \sin(g_2 t + \alpha_2)}{N_1 \cos(g_1 t + \alpha_1) + N_2 \cos(g_2 t + \alpha_2)};$$

therefore

$$\frac{2\,\dfrac{d\varpi_1}{dt}}{\sin 2\varpi_1} = \frac{g_1 N_1^2 + (g_1+g_2) N_1 N_2 \cos\{(g_1-g_2)\,t + \alpha_1 - \alpha_2) + g_2 N_2^2)}{e_1^2 \sin\varpi_1 \cos\varpi_1};$$

therefore

$$e_1^2 \frac{d\varpi_1}{dt} = g_1 N_1^2 + g_2 N_2^2 + (g_1 + g_2)\, N_1 N_2 \cos\{(g_1 - g_2)\,t + \alpha_1 - \alpha_2\}.$$

If $g_1 N_1^2 + g_2 N_2^2$ do not lie between $\pm (g_1 + g_2)\, N_1 N_2$, the apsidal line moves constantly in one direction.

Let $$(g_2 - g_1)\,t + \alpha_2 - \alpha_1 = \psi,$$

$$\frac{d\varpi_1}{dt} = \frac{g_1 N_1^2 + g_2 N_2^2 + (g_1 + g_2)\, N_1 N_2 \cos\psi}{N_1^2 + N_2^2 + 2 N_1 N_2 \cos\psi}$$

is the angular velocity of the apsidal line $= \omega$, suppose

$$\frac{d\omega}{\omega d\psi} = \sin\psi \left\{\frac{2N_1 N_2}{N_1^2 + N_2^2 + 2N_1 N_2 \cos\psi} - \frac{(g_1 + g_2)\, N_1 N_2}{g_1 N_1^2 + g_2 N_2^2 + (g_1 + g_2)\, N_1 N_2 \cos\psi}\right\}.$$

therefore

$$\frac{d\omega}{dt} = -\frac{N_1 N_2 (g_2 - g_1)^2 (N_1^2 - N_2^2) \sin\psi}{(N_1^2 + N_2^2 + 2N_1 N_2 \cos\psi)^2}.$$

Hence, when the eccentricity is the greatest and least, the motion of the apsidal line is fastest or slowest, different arrangements occurring according to the sign and magnitude of N_1 and N_2.

46. In the case of Jupiter disturbed by Saturn, taken as before, N_1 is positive, and N_2 negative, and $N_1 > - N_2$, when ψ

increases through zero, e_1 passes its least value, and $\frac{d\omega}{dt}$ changes sign from $-$ to $+$; therefore ω is a minimum,

$$= \frac{(N_1 + N_2)(g_1 N_1 + g_2 N_2)}{e_1^2}$$

which is negative, since $-g_2 N_2 > g_1 N_1$;

$$\text{for} \quad N_2 : N_1 :: -4 : 11$$

$$\text{and} \quad g_2 = 6g_1 \text{ nearly.}$$

To find the angles through which the Apsidal Line Regredes and Progredes.

47. The apsidal line is stationary when

$$g_1 N_1^2 + g_2 N_2^2 + (g_1 + g_2) N_1 N_2 \cos\psi,$$

and if τ be the least positive angle, whose cosine is

$$-\frac{g_1 N_1^2 + g_2 N_2}{(g_1 + g_2) N_1 N_2},$$

$$e_1^2 \frac{d\varpi_1}{dt} = (g_1 + g_2) N_1 N_2 (\cos\psi - \cos\tau');$$

therefore if N_2 be of contrary sign to N_1, g_1, g_2 positive,

$$\frac{d\varpi_1}{dt} \text{ is negative when } \cos\psi > \cos\tau,$$

or since τ is between 0 and 90°, when ψ increases from $r360° - \tau$ to $r360° + \tau$, r integral, in which case ϖ_1 is decreasing and the apsidal line regredes.

Therefore the apsidal line regredes from the time

$$t = \frac{r360° - \alpha_2 + \alpha_1 - \tau}{g_2 - g_1} = t',$$

$$\text{to} \quad \frac{r360° - \alpha_2 + \alpha_1 + \tau}{g_2 - g_1} = t'',$$

$$t'' - t' = \frac{2\tau}{g_2 - g_1}.$$

If ϖ'', ϖ' be corresponding values of ϖ_1. Let

$$g_1 t' + \alpha_1 = \beta',$$

$$g_2 t' + \alpha_2 = r360° - \tau + \beta'.$$

Then, $$\tan\varpi' = \frac{N_1 \sin\beta' + N_2 \sin(\beta' - \tau)}{N_1 \cos\beta' + N_2 \cos(\beta' - \tau)}$$

$$= \frac{(N_1 + N_2 \cos\tau) \sin\beta' - N_2 \sin\tau \cos\beta'}{(N_1 + N_2 \cos\tau) \cos\beta' + N_2 \sin\tau \cos\beta'};$$

therefore if $$\tan\gamma = \frac{-N_2 \sin\tau}{N_1 + N_2 \cos\tau},$$

$$\varpi' = \beta' + \gamma = g_1 t' + \alpha_1 + \gamma.$$

Similarly $$\varpi'' = g_1 t'' + \alpha_1 - \gamma;$$

therefore $$\varpi' - \varpi'' = g_1 (t' - t'') + 2\gamma$$

$$= 2\gamma - \frac{2g_1\tau}{g_1 - g_2},$$

which is the angle through which the apsidal line regredes, during $\frac{2\tau}{g_2 - g_1}$ years, τ being in seconds.

Again, it progredes from the time

$$t'' = \frac{r360° - \alpha_2 + \alpha_1 + \tau}{g_2 - g_1}$$

to $$t''' = \frac{(r+1)\,360° - \alpha_2 + \alpha_1 - \tau}{g_2 - g_1} = \frac{360°}{g_2 - g_1} + t',$$

$$\varpi''' = g_1 t''' + \alpha_1 + \gamma;$$

$$\therefore \varpi''' - \varpi'' = g_1 (t''' - t'') + 2\gamma$$

$$= \frac{g_1 360°}{g_2 - g_1} - \frac{2g_1\tau}{g_2 - g_1} + 2\gamma$$

which is the angle through which the apsidal line progredes, during $\frac{360.60.60 - 2\tau}{g_2 - g_1}$ years.

48. In the case of Jupiter disturbed by Saturn, employing the numbers given in Art. 44.

$$\tau = 45°\ 25'\ 4'',$$

$$\gamma = 19°\ 10'\ 32'',$$

$$\frac{g_1}{g_2 - g_1}\, 360° = 71°\ 13'\ 48'',$$

$$\frac{g_1}{g_2 - g_1}\, \tau = 8°\ 58'\ 14'';$$

therefore the apsidal line regredes 20° 24′ 36″, in 17456 years,

and progredes 91° 38′ 24″, in 51849 years.

49. The motion may be exhibited by fig. 12, where, at the points *a*, the eccentricity is the least, and the apsidal line retrograding most rapidly; at the points *b*, the eccentricity is greatest, and the apsidal line advancing most rapidly; at the points *c* and *d*, the apsidal line is stationary.

In the case of Saturn disturbed by Jupiter.

$$N_1 = 0{\cdot}035 \text{ and } N_2 = 0{\cdot}048 \text{ nearly};$$

therefore $\frac{d\varpi_1}{dt}$ is always positive,

since $$g_1 N_1^2 + g_2 N_2^2 > (g_1 + g_2) N_1 N_2;$$

therefore the apsidal line of Saturn's orbit moves constantly in one direction.

The figure can be easily constructed by the student for the values of N_1, N_2 given above.

General Relation between the Eccentricities of the Orbits of m *and* m′ *independent of the time.*

50. Since $e_1^2 = N_1^2 + N_2^2 + 2N_1N_2 \cos\{(g_2 - g_1)t + \alpha_2 - \alpha_1\}$,

and $$e_1'^2 = N_1'^2 + N_2'^2 + 2N_1'N_2' \cos\{(g_2 - g_1)t + \alpha_2 - \alpha_1)\},$$

and $[(\gamma - \gamma')D_1 + \sqrt{\{(\gamma - \gamma')^2 D_1^2 + 4\gamma\gamma' D_2^2\}}]\,N_1 + 2\gamma' D_2 N_1' = 0,$

$$[(\gamma - \gamma')D_1 - \sqrt{\{(\gamma - \gamma')^2 D_1^2 + 4\gamma\gamma' D_2^2\}}]\,N_2 + 2\gamma' D_2 N_2' = 0;$$

therefore $$4\gamma\gamma' D_2^2 N_1 N_2 = -4\gamma'^2 D_2^2 N_1' N_2';$$

therefore $$\gamma N_1 N_2 + \gamma' N_1' N_2' = 0;$$

therefore $$\gamma e_1^2 + \gamma' e_1'^2 = \text{constant},$$

or $$m\sqrt{(a)}\, e_1^2 + m'\sqrt{(a')}\, e_1'^2 = \text{constant},$$

which is only a particular case of a more general theorem, which will be established for the case in which all the planets mutually disturb each other's motions.

Example of the Annual Variations of Eccentricity and Perihelion.

51. As an example of the annual variations, the student can calculate the disturbances of Uranus and Neptune by the principal disturbing planets.

In making these calculations, the expressions for $\frac{de_1}{dt}$ and $\frac{d\varpi_1}{dt}$ contain D_1 and D_2, which are the coefficients of $\cos\phi$ and $\cos 2\phi$ in the expansion of $(a^2 - 2aa' \cos\phi + a'^2)^{-\frac{3}{2}}$, and if $b_1^{(\frac{3}{2})}$, $b_2^{(\frac{3}{2})}$ be the corresponding values in the expression of $(1 - 2\alpha \cos\phi + \alpha^2)^{-\frac{3}{2}}$,

$$D_1 = \frac{1}{a'^3} b_1^{(\frac{3}{2})}, \quad D_2 = \frac{1}{a'^3} b_2^{(\frac{3}{2})},$$

and
$$\frac{de_1}{dt} = \frac{m'}{4\mu} n\alpha^2 b_2^{(\frac{3}{2})} e_1' \sin(\varpi_1' - \varpi), \text{ \&c.},$$

$\frac{m'}{\mu}$, $b_2^{(\frac{3}{2})}$, $\log\alpha$, and n, expressed in seconds per year, will be found in Astronomical Tables. See *Pontécoulant* or *Le Verrier*, *Annales de l'observatoire imperial de Paris.*

It must be remembered in employing these variations to find the longitude of perihelion at a given time after the epoch, the precession of the equinoxes must be added.

The results will be found nearly as follows:

Orbit of Uranus.

Annual Variations of Eccentricity in Seconds,

By Jupiter	− 0″·0061
Saturn	− 0 ·0460
Neptune	− 0 ·0024
	− 0 ·0545

Annual Variations of Perihelion,

By Jupiter	1″·25
Saturn	1 ·19
Neptune	0 ·47
	2 ·91

Orbit of Neptune.

Annual Variations of Eccentricity in Seconds,

By Jupiter	− 0″·0011
Saturn	+ 0 ·0031
Uranus	+ 0 ·0094
	+ 0 ·0114

Annual Variations of Perihelion,

By Jupiter	$+0''\cdot006$
Saturn	$-0\ \cdot174$
Uranus	$+0\ \cdot954$
	$0\ \cdot786$

On the Secular Variations of the Inclinations and Longitudes of the Nodes of the Planes of the Orbits.

52. The general equations for determining the variations of the inclinations and longitudes of the nodes of the orbits are

$$\frac{di_1}{dt} = -\frac{n_1a_1}{\mu\sqrt{(1-e_1^2)}}\left\{\frac{1}{\sin i_1}\frac{dR}{d\Omega_1} + \tan\frac{i_1}{2}\left(\frac{dR}{d\epsilon_1} + \frac{dR}{d\varpi_1}\right)\right\},$$

and
$$\frac{d\Omega_1}{dt} = \frac{n_1a_1}{\mu\sqrt{(1-e_1^2)}}\frac{1}{\sin i_1}\frac{dR}{di_1}.$$

If in these equations we write F for R and observe that

$$\frac{dF}{di_1} = -\frac{m'a_1a_1'}{4}D_1\{\tan i_1 - \tan i_1'\cos(\Omega_1'-\Omega_1)\},$$

and
$$\frac{dF}{d\Omega_1} = \frac{m'a_1a_1'}{4}D_1\tan i_1\tan i_1'\sin(\Omega_1'-\Omega_1),$$

we have, preserving only terms of the first order,

$$\frac{di_1}{dt} = -\frac{m'na^2a'}{4\mu}D_1\tan i_1'\sin(\Omega_1'-\Omega_1),$$

$$\tan i_1\frac{d\Omega_1}{dt} = -\frac{m'na^2a'}{4\mu}D_1\{\tan i_1 - \tan i_1'\cos(\Omega_1'-\Omega_1)\}.$$

For $m\sqrt{(a)}$ and $m\sqrt{(a')}$ writing γ and γ' and $na^{\frac{3}{2}} = \sqrt{(\mu)}$,

$$\frac{di_1}{dt} = -\frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1\tan i_1'\sin(\Omega_1'-\Omega_1),$$

$$\tan i_1\frac{d\Omega_1}{dt} = -\frac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1\{\tan i_1 - \tan i_1'\cos(\Omega_1'-\Omega_1)\}.$$

Let
$$p = \tan i_1\sin\Omega_1, \quad p' = \tan i_1'\sin\Omega_1',$$
$$q = \tan i_1\cos\Omega_1, \quad q' = \tan i_1'\cos\Omega_1'.$$

Therefore
$$\frac{dp}{dt} = \frac{di_1}{dt}\sin\Omega_1 + \tan i_1\cos\Omega_1\frac{d\Omega_1}{dt}$$

$$= \tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1 (q'-q),$$

$$\frac{dq}{dt} = \frac{di_1}{dt}\cos\Omega_1 - \tan i_1 \sin\Omega_1 \frac{d\Omega_1}{dt}$$

$$= \tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1 (p-p').$$

Similarly, $$\frac{dp'}{dt} = \tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma D_1 (q-q'),$$

$$\frac{dq'}{dt} = \tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma D_1 (p'-p).$$

Particular integrals of these equations are

$$p = P\sin(gt+\alpha), \quad p' = P'\sin(gt+\alpha),$$
$$q = P\cos(gt+\alpha), \quad q' = P'\cos(gt+\alpha).$$

Therefore, substituting in either of the first two equations

$$gP = \tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1 (P'-P),$$

and from the third or fourth,

$$gP' = \tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma D_1 (P-P');$$

therefore $$\left\{g+\tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma' D_1\right\}\left\{g+\tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}\gamma D_1\right\} = \frac{aa'}{16\mu}\gamma\gamma' D_1^2,$$

or $$g^2 + \tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}(\gamma+\gamma') D_1 g = 0;$$

therefore $$g = 0 \quad \text{or} \quad -\tfrac{1}{4}\sqrt{\left(\frac{aa'}{\mu}\right)}(\gamma+\gamma') D_1;$$

therefore $$P = P' \quad \text{or} \quad \gamma P + \gamma' P' = 0.$$

Hence, if α_1, α_2, and P_1, P_2 be values of α and P corresponding to the roots g and 0, we have

$$p = P_1 \sin(gt+\alpha_1) + P_2 \sin\alpha_2,$$
$$q = P_1 \cos(gt+\alpha_1) + P_2 \cos\alpha_2,$$

and similar expressions for p' and q'.

The constants being determined from the values i, Ω, i', Ω', at the commencement of the epoch, by the equations

$$\tan i \sin \Omega = P_1 \sin \alpha_1 + P_2 \sin \alpha_2,$$
$$\tan i \cos \Omega = P_1 \cos \alpha_1 + P_2 \cos \alpha_2,$$
$$\tan i' \sin \Omega' = -\frac{\gamma}{\gamma'} P_1 \sin \alpha_1 + P_2 \sin \alpha_2,$$
$$\tan i' \cos \Omega' = -\frac{\gamma}{\gamma'} P_1 \cos \alpha_1 + P_2 \cos \alpha_2,$$

and where these constants have been determined

$$\tan^2 i_1 = P_1^2 + P_2^2 + 2P_1P_2 \cos (gt + \alpha_1 - \alpha_2),$$
$$\tan \Omega_1 = \frac{P_1 \sin (gt + \alpha_1) + P_2 \sin \alpha_2}{P_1 \cos (gt + \alpha_1) + P_2 \cos \alpha_2},$$

and similarly for $\tan^2 i_1'$ and $\tan \Omega_1'$.

(*To be continued.*)

THE END OF VOL. II.

ERRATA AND ADDENDA.

Page 306, lines 5 and 6 from the bottom, *read*

$$\left.\begin{array}{l} x = \pm(-1)^n b \cos\{(2n \pm 1)\,a - \theta\} \\ y = \mp(-1)^n b \sin\{(2n \pm 1)\,a - \theta\} \end{array}\right\}.$$

Page 307, the fourth paragraph should be expunged.

Page 308, insert the following as a note at the bottom:

When the Caustic is wholly within, there may be drawn $2n - 1$ tangents parallel to x and terminated by the reflector, and it may be inferred from (20) and (21) that the sum of n of these is equal to the sum of the remaining $n - 1$. There are also $2n$ tangents parallel to y, and the sum of the distances from the centre of n of these is equal to that of the remaining n. It appears too, from (20), that a cubic may be very simply solved by the second caustic; for if $x^3 - qx + r = 0$ and b be taken $= \sqrt{\left(\frac{q}{2}\right)}$ and $a = \frac{q^2}{4r}$, the tangents parallel to x are the roots of the cubic.

Page 309, line 7, *read*

$$2n - \tan 2na \cot a.$$

Page 309, line 20, *read*

$$2n + \cot 2na \cot a.$$

Page 309, line 22, *read*

$$2n + \cot 2na \cot a = \frac{1 + 3\tan^2 a}{2\tan^2 a}.$$

Page 310, last line but one, *read*

$$x = -(-1)^n a \cos(2na - 2\theta).$$

CAMBRIDGE:
PRINTED BY W. METCALFE, TRINITY STREET, CORNER OF GREEN STREET.

Fig. 2.

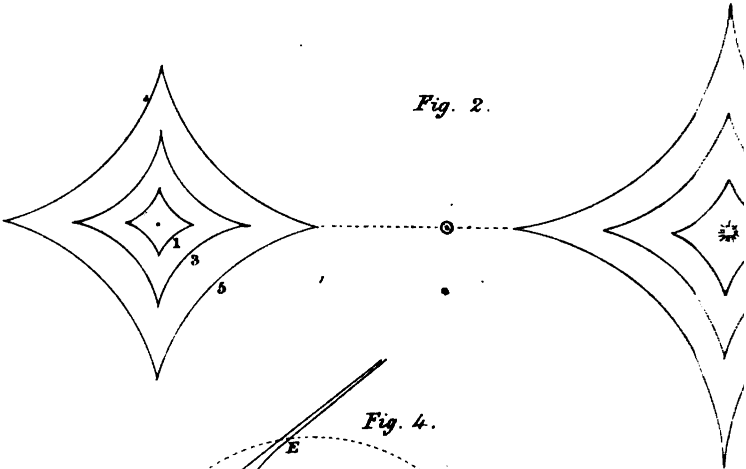
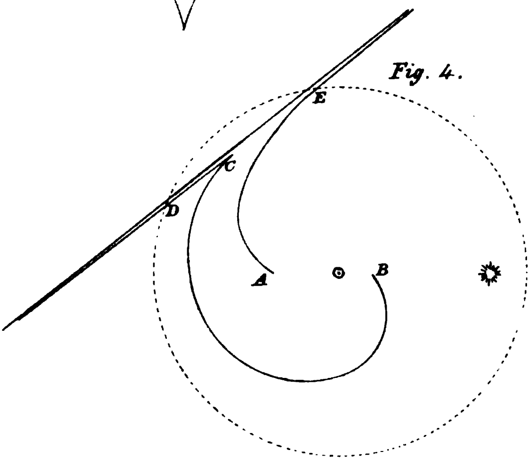


Fig. 4.



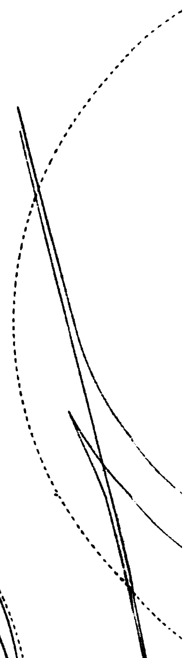


Fig. 6.

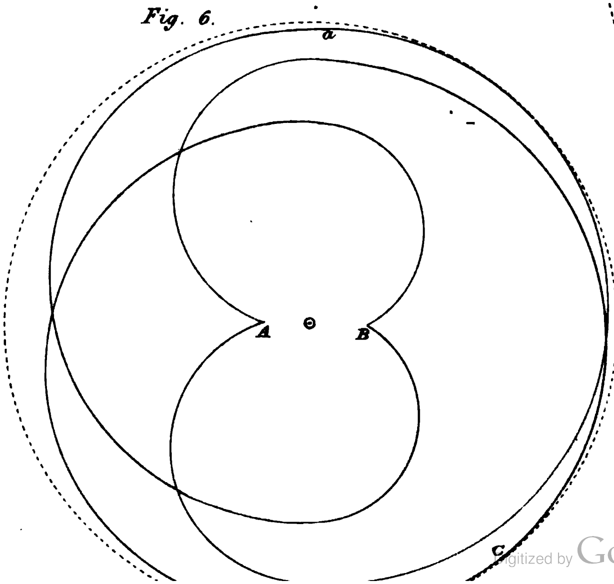


Fig. 7

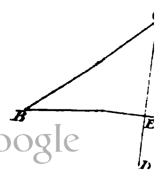
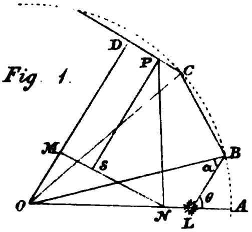


Fig. 1.



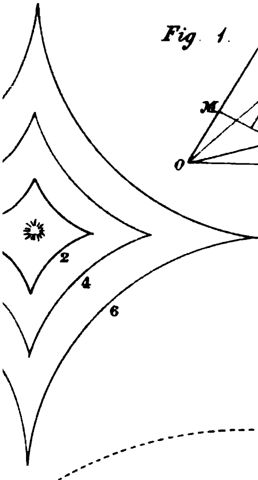


Fig. 3

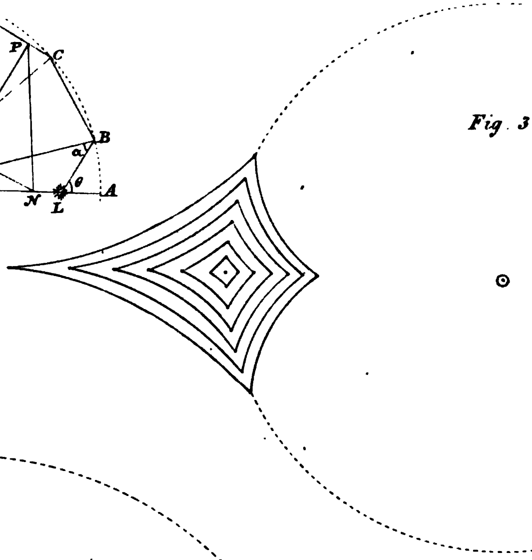


Fig. 5.

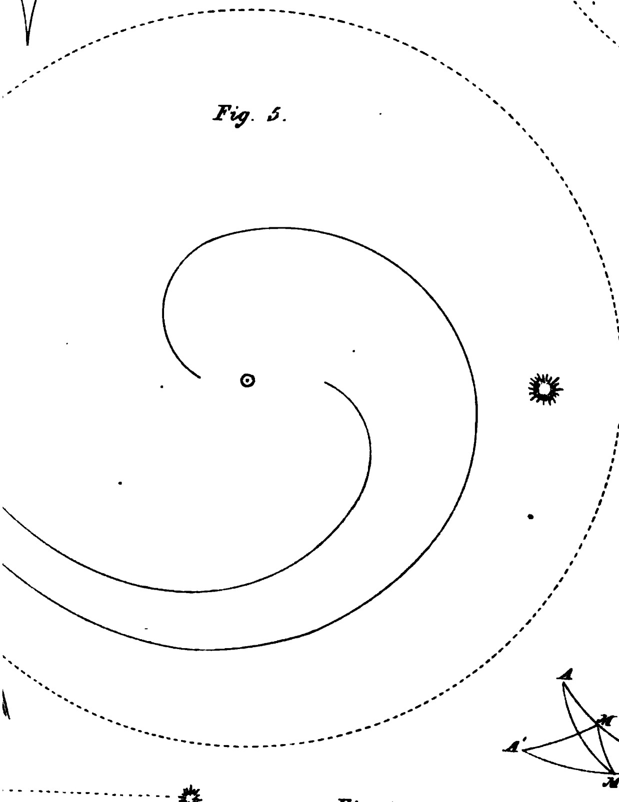


Fig. 10.

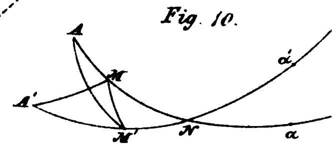


Fig. 8.

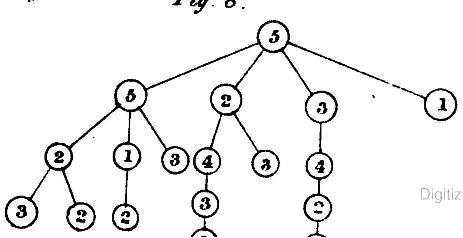


Fig. 7.

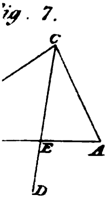
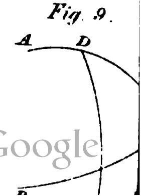


Fig. 9.



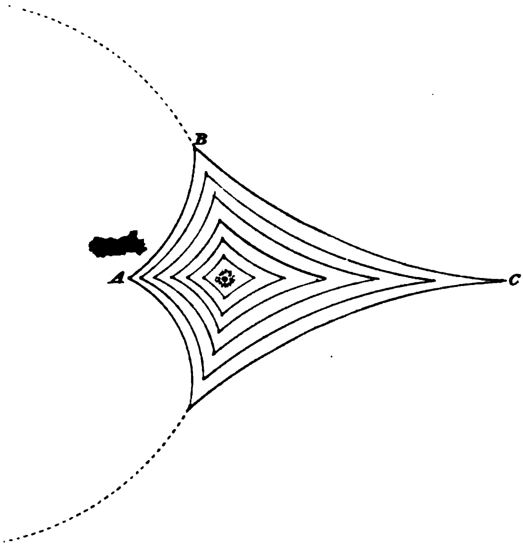


Fig. 11.

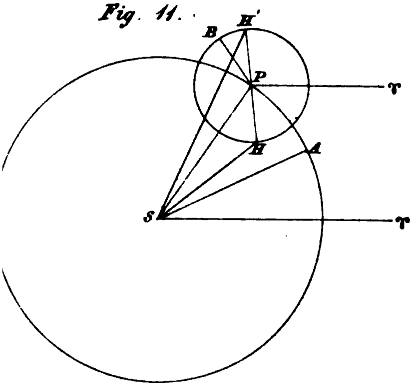


Fig. 12.