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that he was almost certain to make mistakes in going over a piece of work for the first time, and that he could generally not find his errors until he would work over the problem a second time using different symbols. He frequently got very enthusiastic over the theory of groups remarking (not very seriously) that everything could be done by means of groups.

The students talked a great deal about his peculiarities but they had great respect for his attainments. In speaking about Klein and Lie, a Frenchman who had studied under both remarked, "Klein is a gentleman while Lie is just a good fellow." I presume many of those who studied under both would agree that this statement conveys a great deal of truth.

I feel encouraged to publish these few personal observations because I believe that Lie would have liked to have his faults go with his merits. I do not think he wanted to be regarded as one who thought that he had mastered every part of the extensive science of mathematics or as one who thought that all his habits were exemplary. He believed that he had contributed materially towards the advancement of the science of mathematics and he worked hard to accomplish this end. Future developments along the lines which have been emphasized or opened by him will have a great influence on his relative position among the mathematicians of this century.

Cornell University, June, 1899.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLERHE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

Grassmann's Ausdehnungslehre is one of the few great works of mathematics of the 19th century. Appearing first in 1844 and rewritten in 1862, it is only within the last decade or two that it has received a tardy recognition. One reason for this is found in the difficulty of the subject itself, being unlike other mathematics; and another, in the rigorous methods of presentation adopted by the author. In the Ausdehnungslehre of 1862, following some 150 pages of theory, the author for the first time gives the subject concrete form by applying his method to geometry. The theoretical part is naturally the more difficult, while the application to geometry is the more interesting. Hyde, in his Directional Calculus, purposing to present the Ausdehnungslehre to American readers, cut the knot of the difficulty by taking the results of the theoretical part for granted and giving only the application to geometry, and by limiting his treatment to two and three dimensional space.

An elementary exposition which will give the simpler portions of the theoretical part as well as the applications of the theory seems to be needed.
Such an exposition should serve the needs of two classes of readers: First, of those who would like to have a good general idea of the subject without going very deeply into its particulars; and secondly, of those who, expecting to make a thorough study of the subject, wish first to read an introduction to it. To meet this want the following pages have been written. In some places for the purpose of making the subject clearer, changes have been made, but wherever they have been introduced, attention is always called to them.

CHAPTER I.
INTRODUCTION.

1. In elementary mathematics only one kind of unit is admitted, or at most two, viz., 1 and \(-1\). In the Theory of Extension besides the absolute unit of arithmetic and algebra, extensive quantities appear. The extensive quantities are different in nature from the absolute unit and different from each other. As a simple example of an extensive quantity we may name a vector (§3).

2. A Scalar quantity is a quantity of elementary mathematics, i.e., a simple number, either positive or negative.

3. A Vector is a straight line whose length and direction are fixed but not its position. Thus any two parallel and equal straight lines may represent the same vector. A vector gives the relative position of one point with reference to another, viz., a certain distance in a certain direction.

4. Vectors can be added and subtracted.

Thus if \(e_1\) is the vector from \(O\) to \(A\) and \(e_2\) the vector from \(A\) to \(B\), the sum of \(e_1\) and \(e_2\) is \(e_3\), because translation from \(O\) to \(B\) along the straight line \(OB\) is equivalent, or equal in the vector sense, to translation along \(OA\) and \(AB\). Thus,

\[e_1 + e_2 = e_3.\]

Transposing, we get

\[e_1 = e_3 - e_2 = e_3 + (-e_2).\]

Interpreting this equation we see that translation along \(e_3\) followed by translation along \(e_2\) in the negative direction is equal to translation along \(e_1\).

The sum of any number of vectors may evidently be found in the same way. Thus, in the figure

\[e_5 = e_1 + e_2 + e_3 + e_4.\]

Stated generally, we have

The sum of any number of vectors is found by joining the beginning point of the second vector to the end point of the first, the beginning point of the third to the
end point of the second, and so on; the vector from the beginning point of the first vector to the end point of the last is the sum required.

The sum and difference of two vectors are the diagonals of the parallelogram whose adjacent sides are the given vectors.

Or, more explicitly—

(1). The sum of two vectors going out from an origin and forming two sides of a parallelogram is that diagonal of the parallelogram which passes through the origin.

(2). The difference of two such vectors is that diagonal which proceeds from the end of the subtrahend vector to the end of the minuend vector.

5. Vectors and line segments give us simple examples of extensive quantities. We proceed to show how lines and vectors can be used in a system of coordinates.

6. The simplest case of this is where a point is located on a given line. Let \( \rho \) denote any given line, and let \( O \) be an origin on it. Let further \( x \) be a scalar. Then by giving the proper value to \( x \), \( x \rho \) will locate any point \( P \) whatever on the line. Here \( \rho \) is to be regarded as an extensive quantity since it denotes not a number but the position and length of a line.

7. The next simplest case of coordinates is that in which a point is located in a plane by means of two vectors.

Let \( O \), the origin, be a point in the given plane, and \( \varepsilon_1 \) and \( \varepsilon_2 \) two unit vectors in this plane.

Then, by making

\[
\rho = x_1 \varepsilon_1 + x_2 \varepsilon_2
\]

where

\[
x_1 = \frac{\rho \sin \angle BOP}{\sin \angle BOA}, \quad x_2 = \frac{\rho \sin \angle POA}{\sin \angle BOA},
\]

and \( \rho \) = length of \( \rho \), the point \( P \) may be located at any point of the plane. Here \( \varepsilon_1 \), \( \varepsilon_2 \) and \( \rho \) are extensive quantities, and \( x_1 \) and \( x_2 \) are scalars.

8. In space we may have a similar system containing three vectors.

Thus, if

\[
\rho = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3
\]

by assigning values to \( x_1 \), \( x_2 \), and \( x_3 \), \( P \) the extremity of \( \rho \) may be located at any point in space.

9. In the same way we can have a system including four vectors. Thus, if \( P \) is any point, \( \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \) are four unit vectors, and
etc., we have the equation

\[ \rho = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 + x_4 \varepsilon_4, \]

i. e. translation along \( \varepsilon_1 \) a distance equal to \( x_1 \), followed by translation along \( \varepsilon_2 \) a distance equal to \( x_2 \), and so on, is equivalent to translation from \( O \) to \( P \) direct. A geometrical proof of the truth of this can be given, but it is not thought necessary to insert it here.

**Remark.** In the preceding the \( \varepsilon \)'s in a certain sense denote dimensions. Then the space considered in this last article is of the fourth order.

**CHAPTER II.**

**ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF EXTENSIVE QUANTITIES.**

10. **Definition.** A quantity is said to be independent when it can not be expressed in terms of others. A quantity is said to be dependent when it can be numerically expressed in terms of others, i. e. as a sum formed out of numerical multiples of these quantities.

Thus if \( a_1, a_2, \ldots \) are extensive quantities, and \( \alpha_1, \alpha_2, \ldots \) are real numbers, positive or negative, and

\[ a = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \ldots \]

\( a \) is said to be a dependent quantity and to be 'numerically expressed' in terms of \( a_1, a_2, a_3, \ldots \).

11. A quantity \( a_1 \) is a Unit if it can serve along with other like units, \( a_2, a_3, \ldots \) to give a series of numerically derived quantities, \( a, \ldots \). A unit is said to be original if it is not derived from other units. A set of quantities which are independent, i. e., no one of which is numerically expressible in terms of one or more of the others is called a System of Units, provided any number of other quantities can be expressed in terms of them.

As an illustration of such a system we may take the set of units given in Art. 8. There \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are a set of quantities which are independent because any sum formed out of multiples of \( \varepsilon_1 \) and \( \varepsilon_2 \) can never be a quantity like \( \varepsilon_3 \), since any sum formed from \( \varepsilon_1 \) and \( \varepsilon_2 \) would be a quantity in the plane of these two (7) while \( \varepsilon_3 \) is outside of this plane. Moreover, any number of other quantities, \( \rho \)'s, can be derived from \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \).

12. **Definition.** An Extensive Quantity is a quantity numerically derived from a system of units. If an extensive quantity can be derived from the original units it is called an extensive quantity of the first kind.

13. **Definition.** Quantities from the same system can be added (or subtracted) by adding the numerical coefficients of the same units.
Thus

$$\Sigma \alpha e + \Sigma \beta e = \Sigma (\alpha + \beta)e$$

where the $\alpha$'s and $\beta$'s under the summation signs are numbers, and the $e$'s are extensive units. We may add here that in the *Ausdehnungslehre* the distributive law is always assumed to hold.

**Remark.** To remove ambiguity it will be understood that all indicated operations are performed as one comes to them from the left. Thus $a+b+c$ means $(a+b)+c$, and $abc$ means $(ab)c$.

14. The following formulas underlie and justify all the operations involved in addition and subtraction in algebra. They follow directly from the definition in 13.

1. $a+b=b+a$, a commutative law in addition and subtraction.
2. $a+(b+c)=a+b+c$, associative law in addition and subtraction.
3. $a+b-b=a$, opposite character of addition and subtraction.
4. $a-b+b=a$, hence all the laws for addition and subtraction of algebraic numbers hold also for extensive quantities.

15. **Definition.** When an extensive quantity is multiplied (or divided) by a number each of its coefficients is multiplied by that number.

Thus

$$\Sigma \alpha e \cdot \beta = \Sigma (\alpha \beta)e.$$  

**Remark.** If $a$ is an extensive quantity and $\alpha$ a number, then in $\alpha a$ or $aa$ the numerical factor is the multiplier and the other factor is the multiplicand.

16. From Art. 15 we infer the following formulas:

1. $a \alpha = a \alpha$,
2. $a \beta \gamma = a (\beta \gamma)$,
3. $(a+b) \gamma = a \gamma + b \gamma$,
4. $a (\beta + \gamma) = a \beta + a \gamma$,

where, as heretofore, the Greek letters denote real numbers and the Roman letters, extensive quantities. From these formulas it follows—

That all the laws of multiplication and division of algebraic quantities hold also for extensive quantities multiplied or divided by numbers.

17. **Definition.** The totality of quantities which are derivable from a series of extensive quantities, $a_1, a_2, a_3, \ldots, a_n$ is called the Space of those quantities. A space which can be formed out of not less than $n$ such quantities each of the first kind (12) is called a space of the $n$th order.

18. **Definition.** If every quantity of a space ($A$) is at the same time a quantity of another space ($B$), while the converse is not true, then the spaces are said to be incident; the first is said to be subordinate to the second, and the second, to include the first.

19. **Definition.** If $n$ independent quantities $a_1, \ldots, a_n$ can be numerically expressed in
terms of \( n \) other quantities \( b_1, \ldots, b_n \), then is the space of the first quantities identical with that of the last quantities. But if the \( n \) quantities \( a_1, \ldots, a_n \) can be expressed in terms of less than \( n \) quantities \( b_1, \ldots, b_n \), then \( a_1, \ldots, a_n \) are not independent, and some of them can be numerically expressed in terms of others.

20. Two quantities of a space of the \( n \)th order are equal to each other when and only when their numerical coefficients of the same units are equal. This is analogous to the algebraic theorem which says that two complex numbers are equal only when their real parts are equal and also their imaginary parts.

21. If the coefficients \( x_1, \ldots, x_n \) by which an extensive quantity \( x \) is expressed in terms of the units \( e_1, \ldots, e_n \) satisfy an equation of the \( m \)th degree \( f(x_1, \ldots, x_n) = 0 \), then the coefficients \( y_1, \ldots, y_n \) by which \( x \) is expressed in terms of \( a_1, \ldots, a_n \) of the same space also satisfy an equation of the \( m \)th degree, and if the first equation is homogeneous, the latter is also.

Proof. Let \( a_1 = \Sigma \alpha_1 e_r, \ldots \). Then we have

\[
x_1 e_1 + x_2 e_2 + \ldots + y_1 \Sigma \alpha_1 e_r + y_2 \Sigma \alpha_2 e_r + \ldots = \Sigma y_1 \alpha_r e_1 + \Sigma y_2 \alpha_r e_2 + \ldots
\]

\[
\therefore x_1 = \Sigma y_r \alpha_{r1}, \ x_2 = \Sigma y_r \alpha_{r2}, \ldots \quad \text{(Art. 20).}
\]

But if these values are substituted in \( f(x_1, \ldots, x_n) = 0 \), we get an equation of the \( m \)th degree in \( y_1, y_2, \ldots \), and, indeed, homogeneous if the first equation is homogeneous.

[To be Continued.]

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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.


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I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville Tenn., and the PROPOSER.

\[
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\]

\[
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\]